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Learning to “See” Less Than Nothing: Putting Perceptual Skills to Work for Learning Numerical Structure

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How can children’s natural perceptuo-motor skills be harnessed for teaching and learning mathematical structure? We address this question in the case of the integers. Existing research suggests that adult mental representations of integers recruit perceptuo-motor functionalities involving symmetry. Building on these findings, we designed a hands-on curriculum that emphasizes symmetry to teach integer concepts to fourth graders. Compared to two control conditions, children who went through the experimental curriculum showed evidence of incorporating symmetry into their mental representations of integers and performed higher on problems beyond the scope of instruction, including negative fractions and algebra-readiness problems. Gains did not come at the expense of basic integer computation skill. This study has direct practical implications, as current integers curricula generally omit symmetry. The research demonstrates an approach to designing instruction that involves identifying perceptuo-motor functionalities underlying numerical cognition and creating learning activities to recruit them.

Our experiences and actions impact our thoughts. For example, smiling makes us actually feel happier (Kraft & Pressman, 2012; Strack, Martin, & Stepper, 1988); moving fraction manipulatives around as groups helps students construct ideas of part–whole relationships (Martin & Schwartz, 2005). Our human capacities for action and perception also shape our thoughts, by enabling how we experience and act on the world. For instance, brain systems evolved for making comparisons of physical magnitudes enable people to think about comparative size, loudness, and even magnitude of symbolic numbers (e.g., Cohen Kadosh & Henik, 2006; Pinel, Dehaene,

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Riviere, & Le Bihan, 2001). How can instructional designers integrate these ideas to teach mathematics concepts, which are enabled by our perceptuo-motor capacities on the one hand and learned via perceptuo-motor experiences on the other?¹

To gain traction on the matter, we find it useful to consider two subquestions. The *genesis question* asks which perceptuo-motor capacities support mathematical concepts. For instance, negative numbers introduce new structure to the natural numbers in the form of the additive inverse property: $X + -X = 0$. Are there perceptuo-motor systems that allow us to experience this mathematical structure? The *instructional question* asks how we can design activities to recruit these perceptuo-motor systems to help people learn mathematical concepts better. For instance, does jumping a character back and forth on a number line help students learn about negative numbers better than a purely symbolic approach?

Here we address both questions in the case of children learning the integers—positive and negative whole numbers and zero. We review behavioral and neuroimaging evidence that the meaning of the additive inverse property of the integers ($X + -X = 0$) arises from basic perceptual infrastructure that enables people to perceive symmetry. This addresses the genesis question. We then introduce the current study to address the instructional question. We describe a curriculum developed to teach integer symmetry explicitly, and we compare it to two control conditions based on standard models of integer instruction. We show that emphasizing symmetry in integers instruction helps children develop a mental model of negative numbers that subsequently can be used to understand new problems. We conclude with a discussion of the nature of perceptuo-motor driven conceptual change and suggestions for the design of hands-on materials.

PERCEPTUO-MOTOR CAPACITIES FOR MATHEMATICS COGNITION

Before turning to integer cognition, in this section we describe our guiding theory on the relation between mathematical thinking and human perceptuo-motor capacities/neural systems. First, we clarify that by perceptuo-motor systems, we refer to neural systems that are responsible for taking in and representing perceivable information. Second, and more to the point, we take mathematical concepts as rooted in these evolutionarily endowed perceptuo-motor systems, which lend structure to a given mathematical concept (cf. Gibson & Gibson, 1955). These systems are responsible for encoding and comparing inputs along specific dimensions, and they enable people to understand the dimensions in stimuli perceived in the world. For example, to experience one object being heavier than another object, there needs to be a perceptuo-motor substrate that is capable of encoding and comparing weights. The perceptuo-motor system for symmetry allows people to relate visual images on a continuous scale by how symmetric they are.

In mathematics cognition, the most well researched perceptuo-motor contributor is the ability to perceive and compare magnitudes. Numerous behavioral and neuroimaging studies suggest that the neural systems responsible for perceptual magnitudes, which are evolutionarily old and similar across species, get recruited to support the culturally constructed number system. Moyer

¹We adopt the expression “perceptuo-motor” rather than perception or action, because we consider these two are inextricably linked, and referring to motor action without an appreciation of perception is intractable and vice versa (Gibson, 1979; Gibson, 1988).

and Landauer's (1967) seminal study asked participants to compare pairs of natural numbers presented as symbolic digits by judging which was lesser (greater). Natural numbers are positive whole numbers. They found that comparing symbolic digits resulted in similar reaction time patterns as are found when comparing perceptual magnitudes, such as the loudness of tones or quantities of dots. Subsequent neuroimaging research has identified a region of the brain that appears to be responsible for both perceptual comparisons of magnitude and symbolic comparisons of digit magnitudes (intraparietal sulcus, e.g., Cohen Kadosh et al., 2005; Fias, Lammertyn, Reynvoet, Dupont, & Orban, 2003; Holloway, Price, & Ansari, 2010; Piazza, Pinel, Le Bihan, & Dehaene, 2007; Pinel et al., 2001). Hence, even though the natural number system—the digits—is a human cultural construction, understanding numerical magnitude seems to stem from the native perceptual ability to compare magnitudes. Dehaene and Cohen (2007) have referred to the process of exapting an evolutionarily bestowed ability to serve a new cultural function as “cortical recycling.” By this account, the perceptuo-motor functionality enables and limits mathematical understanding (e.g., Dehaene & Cohen, 2007; Lipton & Spelke, 2005).

To understand other properties of number, such as ordinality and cardinality, Schwartz, Blair, and Tsang (2012) propose that people recruit other perceptuo-motor systems, such as the ordering ability of motor plans and the visual ability to identify two versus three objects without counting (i.e., subitizing). These distinct systems are bundled together to create a full concept of natural number. Figure 1 is a schematic for integration of different contributing perceptuo-motor systems, called the Bundling Hypothesis (Schwartz et al., 2012). The Bundling Hypothesis proposes that integrating perceptuo-motor capacities ensures that children learn that 5 big, 5 total, and 5th have a tight relation and that a change within one type of meaning (e.g., five objects becomes six objects) entails changes within the other types (6 is more than 5).

Such a proposal is consistent with Case and Okamoto's (1996) theory for the developmental of central conceptual structures, and it has empirical support. Children who went through instruction that focused on integrating order, counting, and magnitude did better in math the following year than students whose curricula emphasized each sense of number separately (Griffin, Case, & Siegler, 1994).

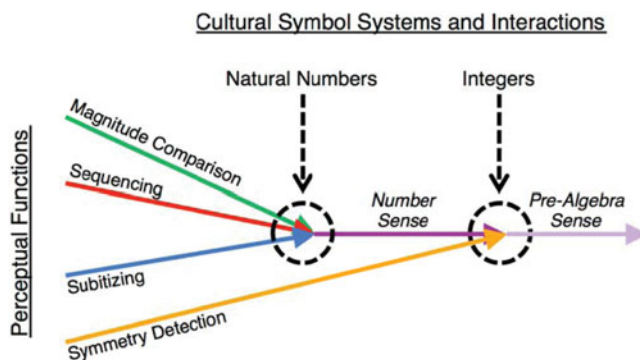


FIGURE 1 Schematic of the integration of perceptuo-motor systems for numerical cognition. Adapted from Schwartz, Blair, and Tsang (2012).

SYMMETRY IN INTEGER COGNITION AND LEARNING

According to the Bundling Hypothesis, learning new mathematical structure beyond the natural numbers involves bringing in additional perceptuo-motor capacities to the existing bundle. For instance, with the introduction of the integers, an additional perceptuo-motor system is recruited to support the new mathematical structure of the additive inverse property. The proposal we consider in this section, as captured in Figure 1, is that integrating the perceptuo-motor capacity of symmetry supports learning the integers. We present evidence for this claim from prior behavioral and neuroimaging research. Then we introduce the current work, a classroom learning experiment designed directly from the prior findings.

Prior Research on Genesis: Symmetry Systems Support Additive Inverse Concepts

The cognition of negative numbers is an active area of research (e.g., Dodd, 2011; Fischer, 2003; Gullick, Wolford, & Temple, 2012; Nuerk, Iversen, & Willmes, 2004; Shaki & Petrusic, 2005; Tzelgov, Ganor-Stern, & Maymon-Schreiber, 2009). For example, Blair, Rosenberg-Lee, Tsang, Schwartz, and Menon (2012) demonstrated that in adults, negative numbers have distinct neural representation from positive numbers. Of special relevance here is work that reveals the role of symmetry. Varma and Schwartz (2011) asked people to compare the magnitude of digits, including positive and negative numbers. They developed several models for how people represented the integers and found that a model that encoded symmetry provided the best fit to the reaction time data. Tsang and Schwartz (2009) sought empirical evidence of symmetry in mental integer representations. In their experiments, subjects saw two digits on a screen and determined the numerical value midway between them. For instance, given -4 and 10 , the midpoint would be 3 . (The numbers were always positioned in the same place on the screen regardless of magnitude.) People were increasingly fast at solving the problems when the digit pair became more symmetric about zero. For example, -6 and 8 was answered more quickly than -4 and 10 . This was true even when people were told to use a specific symbolic computation to solve the problem (i.e., add the two digits and divide by 2). The authors concluded that people were taking advantage of symmetry to help solve the problems. In follow-up neuroimaging work, Tsang, Rosenberg-Lee, Blair, Schwartz, and Menon (2010) found that regions of the brain proximal to those associated with visual symmetry (Sasaki, Vanduffel, Knutsen, Tyler, & Tootell, 2005; Tyler et al., 2005) increase in activation as symbolic integer symmetry increases. Participants exhibited more activation in these regions when they bisected more symmetric intervals compared to less symmetric intervals, even though the visual digits they saw on the screen had no symmetry-related visual patterns. The study raises the possibility that similar brain systems underlie adults' processing of perceptual visual symmetry and conceptual symmetry of the integers.

To summarize, our proposal for the genesis question is that people recruit the distinct perceptuo-motor system of symmetry to make meaning of and to work with integer structure. If true, how can we use this knowledge to help children learn? This brings us to the instructional question.

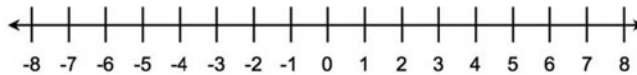


FIGURE 2 Integer number line.

Instructional Proposal: Targeted Perceptuo-Motor Experiences Can Help Students Recruit Symmetry Systems for Integer Understanding

If we accept that perceptuo-motor capabilities support quantitative understanding, the instructional question is how to create instructional activities that recruit and integrate these capacities. Hands-on activities involving manipulatives—touchable materials often used in elementary schools to represent mathematical concepts—seem like a natural fit. Manipulatives, however, are not always effective despite their popularity (Clements & McMillen, 1996; Martin, Lukong, & Reeves, 2007; McNeil & Jarvin, 2007; Uttal, Scudder, & DeLoache, 1997). Additionally, it is unclear how manipulatives can be adapted for learning beyond the natural numbers. As natural numbers describe quantities with which children physically interact from a young age, they are readily instantiated in concrete classroom materials; but as negative quantities are not easily found in the physical world, they present a greater challenge to tangible instantiation. Our framework may help address these issues by suggesting that an important role of hands-on activities is to help focus attention on particular aspects of mathematical structure, which aids in the recruitment of the appropriate perceptuo-motor systems.

Prior research has shown that learning from hands-on activities can be effective when the materials accurately reflect numerical structure (e.g., Siegler & Ramani, 2009), but that this is not always sufficient because students may not notice the critical properties (e.g., Ball, 1992). Thinking about perceptuo-motor contributors to numerical cognition can help curriculum designers generate ways for students to interact with them such that the students come to “see” the numerical structure in the materials. Figure 2 presents an example of a number line. Well-educated adults perceive in it important features shared with the natural numbers including the sequential ordering of the numerals and the equal intervals between each. Adults may further see that -1 and 1 are symmetric about 0 , as are -2 and 2 , and so forth. Children may not easily see these properties (e.g., Saxe et al., 2010). Figure 3 shows a fourth-grade child’s recreation of a number line after learning about integer number lines for several days. The child failed to reproduce key structural properties of the number line—the order of the numbers and, by extension, symmetry across zero—implying that these numerical relationships were not part of his internal representation (Siegler & Booth, 2004; Siegler & Opfer, 2003) and that he did not notice them in the learning materials.

As previous research has implicated the perceptuo-motor capacity of symmetry as supporting additive inverse concepts, the challenge of the current work was to generate ideas for how to get

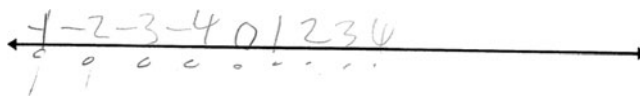


FIGURE 3 Incorrect number line drawn by a fourth grader after several sessions of integer number line activities.

students to notice the symmetry in the integers and build it into their mental representations of integers. Our instructional approach builds on J. J. Gibson's (1979) and E. J. Gibson's (1982) work on affordances and perceptual learning. External representations like the number line have structural affordances that children may or may not perceive immediately; with learning, affordances are revealed. Consistent with a Gibsonian account of perceptual learning and discrimination, we focus first on helping fourth-grade students bring to bear their innate abilities to perceive symmetry to notice structure in physical instantiations of an integer number line. We develop materials that visually highlight integer symmetry about zero and have students engage in actions that make symmetry salient. We also develop supporting activities to help integrate symmetry structure with other structural properties of the number line, like order and magnitude. Going beyond Gibsonian accounts of perceptual learning, the numerical cognition literature proposes that mental representations of natural numbers are spatial in nature (e.g., Hubbard, Piazza, Pinel, & Dehaene, 2005) and not necessarily bound to specific visuo-perceptuo experiences. Accordingly, the hope is that our curriculum's visual and perceptual experiences will help students develop mental representations of integers that partake of this symmetry structure when perceptual stimuli are removed.

The curriculum prompts children to *notice* symmetry in the integers, *integrate* symmetry with other properties of integers, and *internalize* these structures into their mental representations of the integer number system. This process should measurably impact children's thinking with integers, most notably as predicted by mental model theory (Johnson-Laird, 1983). According to the theory, people's mental models, or internalized versions of the structure of the experienced world, drive how they interpret and draw inferences from their surroundings. Mental models that incorporate structural relations and processing borrowed from perceptuo-motor systems will constrain and enable interpretations and reasoning even about situations never directly experienced before. For instance, if a person's mental model of positive numbers accurately incorporates the structure of magnitude, the person will be able to figure out that 244 is greater than 187, even if the person has never before compared those particular numbers. For the integers, we expect our curriculum to lead students to develop mental models of integers that accurately reflect integer structure, including the additive inverse property via symmetry. In turn, we expect these mental models to enable the students to solve basic integer problems and, more importantly, to be generative in problem solving situations that are related to their learning but not directly covered in the curriculum.

Current integer instruction generally does not explicitly emphasize symmetry. Integer symmetry is not a deliverable in the Common Core State Standards, although its importance is implied in standards on numerical opposites and directed magnitude (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). A random sampling of three textbooks on the current list of approved California curricula showed that only one—*Everyday Mathematics* (Bell et al., 2004)—briefly mentions the mirror image between the negatives and positives in its introduction to integers. To our knowledge, only one other experimental integer curriculum has focused on symmetry (Saxe et al., 2010), where it was one of five principles that were taught over two 45-minute sessions. Including symmetry emphasis in integer curricula is thus novel and may offer a new foothold for teaching integer structure. To be clear, in our work we offer symmetry emphasis as a supplement to, rather than a replacement for, current instructional models known to help children make sense of integers. Our instructional goal is to help students learn the structure of the integer number system. Other research has focused on

improving students' sense-making and meaning-making around integers. For example, several design studies aiming to improve integer instruction have focused on providing realistic referents for the meaning of the integers to help students reason about them (e.g., Gregg & Gregg, 2007; Liebeck, 1990; Linchevski & Williams, 1999; Stephan & Akyuz, 2012).

To review, the instructional question we consider is how to design instruction to help students harness their perceptuo-motor capacities like symmetry to learn mathematical structure. Importantly, children already are adept at seeing symmetry in the visual world (Bornstein, Ferdinandsen, & Gross, 1981; Fisher, Ferdinandsen, & Bornstein, 1981) so the goal of instruction is not to help them define symmetry or improve general symmetry sensitivity. Rather, the task is to help students integrate their native symmetry abilities into their mental representations of the number system. We expect that, compared to two control curricula, our symmetry-focused curriculum will lead students to mental integer representations that are more complete and more generative in the face of unfamiliar challenges. If these expectations are realized, they would indicate that symmetry support for the additive inverse is both possible to teach and beneficial to students' mathematical thinking.

EXPERIMENT

We designed three instructional conditions to teach integer semantics and addition to fourth graders. The first two emphasize existing models and methods of integer instruction, and the third supplements existing models by introducing symmetry as a framework for thinking about integers.

The existing instructional models emphasized are the integer number line model and a cancellation model (sometimes called a neutralization model), which are often used either separately or together in integer instruction (Bofferding, 2011, 2014; Stephan & Akyuz, 2012). In the number line model, addition is modeled as movement along the integer number line. For example, students imagine standing on the first addend. Facing in the positive direction and walking forward means adding a positive number, and facing in the negative direction and walking forward means adding a negative number (Swanson, 2010). In the cancellation model, students are taught to think of the negative and positive integers as representing opposite quantities. Students learn that positive and negative quantities cancel each other out, and arithmetic problems are enacted using colored chips. For example, for the problem $-5 + 2$, five negative chips and two positive chips are placed in parallel rows, and the positive-negative pairs cancel out, leaving -3 chips remaining as the answer (e.g., Battista, 1983).

The three conditions in the current study—the jumping, stacking, and folding conditions—use the standard models of instruction in different ways (see Table 1). The jumping condition includes the number line model only. The stacking condition includes both the number line and cancellation models. The folding condition, which is of most interest here, includes both the number line and cancellation models and furthermore emphasizes integer symmetry.

Predictions

As motivated above, we broadly expect that because symmetry supports important mathematical structure of the integers, students who are taught integer symmetry in the folding condition will

TABLE 1
Curricular Foci, by Experimental Condition

Curricular Focus	Condition		
	Jumping	Stacking	Folding
Number line model	✓	✓	✓
Cancellation model		✓	✓
Symmetry			✓

differ systematically from students in the two control conditions. We present four measurable predictions, including one null control prediction. Example measures are shown in Table 2 and described in more detail in the Results section.

TABLE 2
Measures and Performance Predictions

Name	Description	Prediction	Example Problem				
Strategy	Prevalence of symmetry strategy use and its relationship to accuracy	F > S F > J	<i>Solve. You may draw or write to help you.</i> $1 + -3 + -2 + 5 + -1 =$				
Interference	Response time pattern on number line imagination problems (Zero Zone), as indicator of mental representation	Interference effect: F > S F > J					
Generativity	Topics not directly covered in the curricula (accuracy)	F > S F > J	<i>Where do these go on the number line?</i> $-3/7$ $-1\frac{1}{2}$ -2 				
Basic Fluency	Symbolic Fluency: Integer addition and magnitude comparison (accuracy)	F = S = J	$3 + -2 = 1$ $0 > -8$ $-4 + -1 = -5$ $-2 < -1$				
	Properties: Integer order, interval, magnitude, polarity	F = S = J	<i>Write the numbers in order from least to greatest</i> <table style="margin-left: auto; margin-right: 0;"> <tr><td>-1</td></tr> <tr><td>4</td></tr> <tr><td>2</td></tr> <tr><td>-3</td></tr> <tr><td>-5</td></tr> </table>	-1	4	2	-3
-1							
4							
2							
-3							
-5							

Note. F = folding condition; S = stacking condition; J = jumping condition.

Prediction 1. Strategy: Folding Condition Will Choose More Symmetry-Oriented Strategies During Integer Problem Solving

Students will adopt condition-specific strategies for integer problem solving; specifically, we expect students in the folding condition to use symmetry-oriented strategies on integer problems more often than students in the jumping and stacking conditions, even for tasks that had not been modeled during the curriculum. This would be consistent with extant research showing that knowledge of integer principles affects how people represent novel problems (Prather & Alibali, 2008). It would indicate that symmetry emphasis in an integers curriculum shapes how children think about integer problems.

Prediction 2. Interference: Folding Condition Will be Slower to Imagine Nonsymmetric Number Lines

Existing research indicates that salient features of mental representations can cause people to be especially slow on tasks where that feature is irrelevant (e.g., Cohen Kadosh & Henik, 2006; MacLeod, 1991; Pinel, Piazza, Le Bihan, & Dehaene, 2004; Stroop, 1935; Tzelgov, Meyer, & Henik, 1992). Here, if students have integrated symmetry into their mental models of integers, we expect it to interfere with their processing on certain types of integer tasks.

We predict that when asked to quickly imagine integer number lines, students in the folding condition will show a specific deficit compared to other groups. They will be slower to imagine nonsymmetric number lines than symmetric number lines (e.g., slower to imagine -5 to 7 than -6 to 6). The prediction is tested with a computer-based response time task, the Zero Zone (Figure 4). The expected results would indicate that students in the folding condition imagine number lines via a different process than other students, one that favors symmetry structure automatically regardless of its relevance to the task at hand. The results would suggest that symmetry had become incorporated into students' mental models of integers and would demonstrate the curriculum's impact on mental processing. Importantly from a methodological perspective, this finding would indicate that any performance benefits seen in the folding condition on other measures are not simply due to an overall difference in instructional quality, but rather a specific cognitive process that helps performance in some cases but hurts it in others.

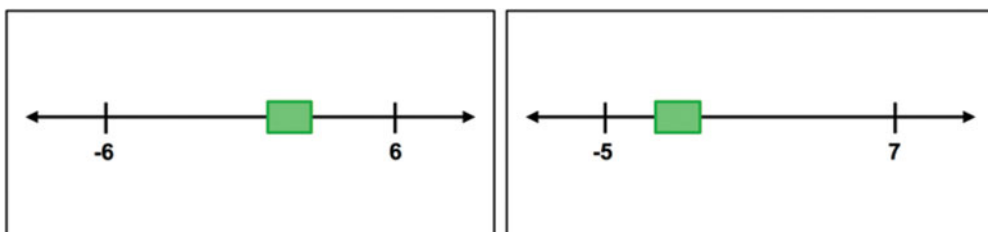


FIGURE 4 Sample trials from Zero Zone interference task. On each trial, students indicated whether or not zero belongs in the green zone. Trials were either symmetric (left panel) or nonsymmetric (right panel).

Prediction 3. Generativity: Folding Condition Will Reason More Effectively About New Mathematical Situations

The third and perhaps most important prediction is that students in the folding condition will perform well on topics beyond the scope of the instruction that are relevant to ideas of symmetry, balancing, and the additive inverse. We measured accuracy on algebra-readiness problems, negative fractions, and novel word problems. If our prediction holds, it would suggest that symmetry emphasis helps children effectively interpret new and unfamiliar mathematical situations. The result would complement prior research showing that characteristics of internal number representations of natural numbers impact learning of new material (Booth & Siegler, 2008).

Control Prediction 4. Basic Fluency: Conditions Will Improve Equally on Symbolic Fluency and Will Successfully Map Natural Number Properties to the Integers

Finally, we predicted that all conditions would learn basic integer fluency equally well, as measured by a posttest on integer addition, magnitude comparisons, and basic relational properties taught to all conditions. During the instruction, all conditions directly learned and practiced the types of problems and properties included on the fluency measure. Thus, the null result would suggest that students in all three instructional conditions paid attention to the lessons and that all conditions met a common basic standard of quality. This result would increase our confidence that other results are not due to gross differences in instructional quality, as it would show that all conditions improved similarly on the instructed content.

Predictions Summary

The predicted data would collectively demonstrate that curricular emphasis on a specific perceptuo-motor-supported structure of the integers can influence students' mental representations and lead to increased generativity in unfamiliar but related situations. Support for the strategy and interference predictions would indicate that the symmetry curriculum led students to integrate symmetry into their mental representations of integers. Support for the generativity prediction would show that this symmetry structure was useful in new mathematical contexts. Support for the basic fluency (null) prediction would reassure us that condition differences on the other measures are not due to an overall dearth of quality in the control curricula.

METHODS

Participants

Seventy-four students participated from three fourth-grade classes at a public school in the San Francisco Bay Area. One class included several third graders. In the year of data collection,

45% of students at the school were English language learners, and 35% were eligible for free or reduced-price lunch.

Students were reassigned from their regular classes to one of three conditions—jumping ($n = 22$), stacking ($n = 26$), or folding ($n = 26$)—matched for gender, grade level, English language proficiency, the previous year's Standardized Testing and Reporting (STAR) scores for mathematics, prior knowledge of integers, and special needs status. The method of condition assignment was stratified random sampling based on STAR scores, plus additional checks to ensure evenness of other demographic statistics. An additional fifteen children (jumping 9, stacking 3, folding 3) were excluded due to absence or lack of parental consent. All classes were represented in all conditions.

Prior to the study the students had not yet learned integers in school.

Procedure

Assessments

Participants took a paper-and-pencil pretest in their original classes several days before the study began. Two days after instruction ended, participants were measured on computer and interview tasks in their condition groups. One or two days later, participants took a paper-and-pencil posttest in their original classes.

Curricular Implementation

Instruction lasted for 4 days, 55 minutes per day, over a period of 3 weeks, and was implemented in a separate room for each condition. Within each condition students were split into two matched subgroups, which participated in the study on alternating days. The instruction was the same between subgroups, but taught by different researchers. This design allowed researchers to instruct small groups of 12–15 students at a time while the nonparticipating subgroups participated in small-group lessons with their regular teachers on other topics.

Three two-person teams of researchers implemented the curriculum. Each team taught two conditions. Teams were assigned to conditions by picking researchers' names out of a hat. All instruction was scripted (see Appendix D for example lesson). Instructors followed the script strictly for lectures, but flexibly for discussions and one-on-one interactions. Flexibility around the script was limited by a list of phrases to avoid (to prevent condition contamination) and a list of phrases to emphasize.

Curriculum

The primary content of the 4-day curriculum was a series of games about integer addition, played with concrete manipulatives (Figure 5) and analogous virtual manipulatives. A typical day included brief whole-class activities or discussions and a longer session of small-group game

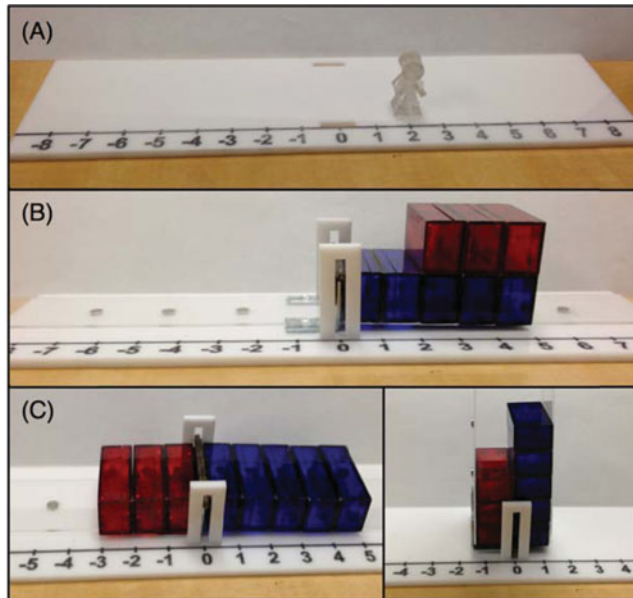


FIGURE 5 Physical materials used for game play. (A) Jumping condition enacted addition by jumping a figurine along a number line. (B) Stacking condition stacked blocks on a number line. Blue blocks designated positive quantities; red block designated negative quantities. (C) Folding condition set blocks on both sides of a number line, and then folded at zero.

play. All conditions played all the same games for roughly the same amount of time. All had the same overall instructional time.

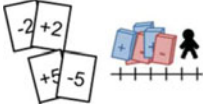


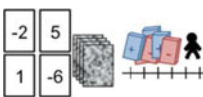

The 4 days of curriculum are summarized in Table 3. Briefly, the curriculum began with an introduction to the negative numbers—what they are and how to write them. Then students played a series of games to practice the core action of their condition (jumping, stacking, or folding) via integers addition within and across the positives and negatives. Short activities and discussions were interspersed to reinforce and supplement in-game learning (Appendix D). Toward the end of the curriculum, students played computer games with virtual manipulatives meant to help them transition from concrete to symbolic representations (number sentences) of integers (McNeil & Fyfe, 2012; Sarama & Clements, 2009).

Following Griffin et al.'s (1994) recommendations for instructional design, instruction was made to be engaging and socially interactive, to integrate integer concepts through concurrent and purposeful use, to use congruent and diverse representations of numbers (symbols, blocks, playing cards, number lines), and to help students bridge between representations such as concrete and symbolic portrayals of integers.

Materials

Materials, lessons, and activities were designed and built by the researchers. The primary manipulatives were a set of plastic number lines, blocks, and figurines (Figure 5) used during game

TABLE 3
Study Schedule

<i>Day</i>	<i>Main Activity</i>	<i>Activity Schematic</i>	<i>Purpose</i>	<i>Other Activities</i>
Pretest	<ul style="list-style-type: none"> • Paper-and-pencil measures (given by teachers) 		<ul style="list-style-type: none"> • Assess prior knowledge • Sample stratification • Covariate measure 	
Day 1	<ul style="list-style-type: none"> • Zero Go Fish! game, two-card pairs 		<ul style="list-style-type: none"> • Vocabulary and visual recognition • Total cancellation/reversing (Additive inverse) 	<ul style="list-style-type: none"> • Introduction to negative numbers • Symmetry framing (folding condition only)
Day 2	<ul style="list-style-type: none"> • Zero Go Fish! game, multi-card "pairs" 		<ul style="list-style-type: none"> • Within-sign addition (neg+neg) 	<ul style="list-style-type: none"> • Magnitude and #line properties activities
	<ul style="list-style-type: none"> • Target Practice game 		<ul style="list-style-type: none"> • Mixed-sign addition (neg+pos) with motion 	
Day 3	<ul style="list-style-type: none"> • Leftovers game with manipulatives 		<ul style="list-style-type: none"> • Mixed-sign addition • Partial cancellation/reversing 	<ul style="list-style-type: none"> • Target Practice game (cont'd) • Activity: Negatives can stand for . . .
Day 4	<ul style="list-style-type: none"> • Leftovers game, computerized – Predict answer – Predict number sentence 		<ul style="list-style-type: none"> • Mass practice with mixed-sign addition • Transition to symbolic operations 	<ul style="list-style-type: none"> • Lesson on imagining manipulatives
Posttest	<ul style="list-style-type: none"> • Computer measures 		<ul style="list-style-type: none"> • Assess mental representation 	
	<ul style="list-style-type: none"> • Videotape measures 		<ul style="list-style-type: none"> • Observe strategy choice 	
	<ul style="list-style-type: none"> • Paper-and-pencil measures (given by teachers) 		<ul style="list-style-type: none"> • Measure learning <ul style="list-style-type: none"> – Symbolic fluency – Properties knowledge – Generativity 	

play. These portrayed key features of the integers. (As noted in the introduction, however, students noticing these features is not a given.) The canonical integer number line, for example, demonstrated compactly the principles of magnitude, order, polarity (positives vs. negatives), and symmetry of the integers, and it allowed students to see negatives as an extension of the

familiar positive number line. Integer features portrayed in blocks and figurines are described in the following section. Additional materials included a set of integer playing cards displaying symbolic (-3) and pictorial representations (dots) of the numbers ±1-6, computer versions of one instructional game, and worksheets and discussion scripts.

Experimental Conditions

There were three experimental instructional conditions: jumping, stacking, and folding. Jumping and stacking were control conditions that mapped onto current instructional models, and folding was the experimental condition that additionally introduced symmetry. The conditions are named for the core action students undertook while physically modeling integer addition: jumping a figurine along a number line, stacking blocks on a number line, or folding the positive and negative sides of a number line together (Figures 5 and 6). These actions called for students to interact with the materials in specific ways to make certain mathematical structure stand out (Lobato, Ellis, & Munoz, 2003).

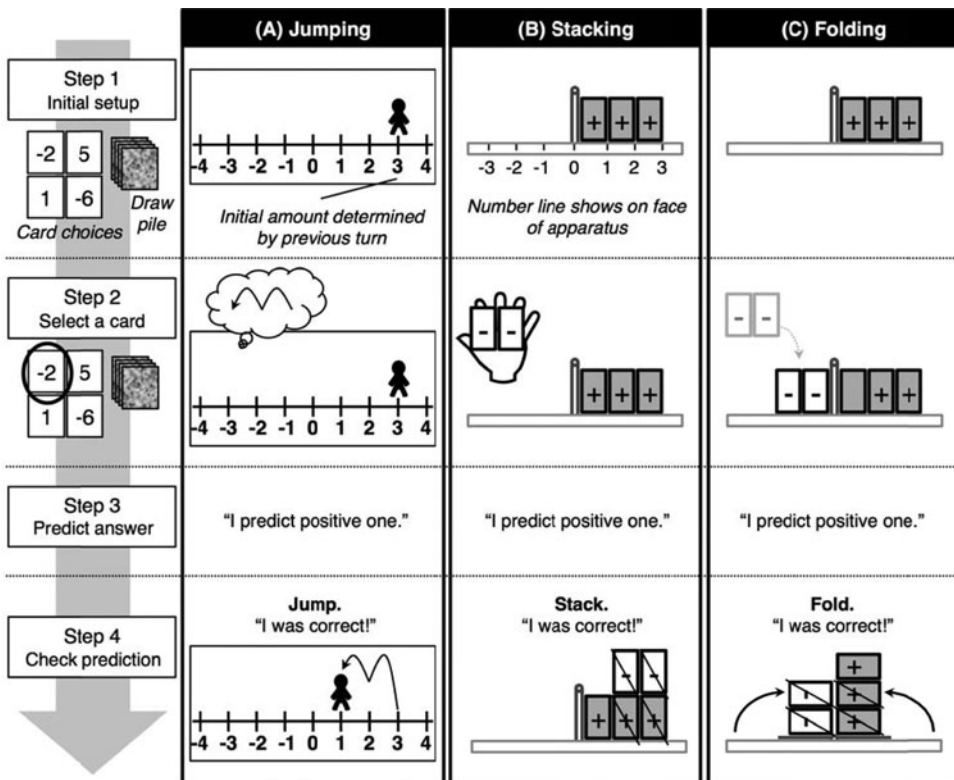


FIGURE 6 Schematic of Leftovers game for each instructional condition. Students played this 2-4-player card game during the second half of the curriculum. Left column shows the steps for one turn in the game. The three panels show the corresponding actions with manipulates for the (A) jumping, (B) stacking, and (C) folding conditions. The example shown is equivalent to solving the equation $3 + -2 = 1$.

Jumping Condition

The jumping condition, which focused exclusively on the number line model of integer instruction, enacted addition problems by hopping a figurine along the number line. Figure 6A shows an example within the context of the Leftovers game. The jumping action supported an interpretation of negative integers and operations as positions and movements. Supplemental activities reinforced the interpretation. For example, students discussed that -3 can stand for 3 feet behind the starting line. For the additive inverse property ($X + -X = 0$), students were taught that positive and negative numbers “reverse back,” a phrase that aligns with the metaphor of arithmetic as movement along a line. The number zero was emphasized as the starting point for positive and negative numbers. Students were not taught symmetry explicitly.

Stacking Condition

In the stacking condition, students primarily used the cancellation model of integer addition. Students enacted addition problems by stacking blocks on the number line. For example, to solve $3 + -2$ (Figure 6B), students placed three blue blocks on the positive side of the number line to represent $+3$. Then, working back toward 0, they placed two red blocks on top to represent -2 . The stacked portion “cancelled out,” and the remaining unstacked portion represented the answer. This use of blocks emphasized amount and categorical distinctiveness of positives and negatives in the integers. For amount, each block represented a unit, but blocks magnetically stuck together in a row to represent quantities greater than one. For categorical distinctiveness, the negative and positive blocks were different colors. While both principles are passively portrayed in a simple integers number line, they were actively emphasized in the stacking condition via the use of the blocks. Students also played one game of jumping a figurine along a number line to experience the number line model of integer addition.

The dual use of cancellation and number line models in the stacking condition was reinforced in supplemental discussions, where negative numbers were portrayed as quantities as well as movements and positions. For example, students were taught that -3 can stand for the amount of money one has when one owes \$3. In this condition, for the additive inverse, negative and positive numbers were described as “cancelling each other out.” Zero was emphasized as the starting point for positive and negative numbers. This condition supplemented the jumping condition with additional ways to think about negatives but did not include explicit symmetry.

Folding Condition

In the folding condition, the number line manipulative was topped with a clear layer of hard plastic that hinged at the number zero (Figure 5C), allowing the students to fold the number line in half. Students represented both addends on the number line at the same time, for example by placing three blue blocks on the positive side and two red blocks on the negative side of zero (Figure 6C). They then folded at the zero hinge to bring the sides together. The blocks that matched up after folding cancelled out, and the ones that did not match up represented the answer.

In this condition, the materials were intended to emphasize all of the properties emphasized in the other conditions, while also portraying the symmetry of the additive inverse by turning zero into a literal inflection point between the positive and negative blocks.

Like the stacking condition, negative numbers were portrayed as quantities as well as movements and positions, and adding positives and negatives was described through cancellation. The folding condition also received a symmetry framing for thinking about the integers. At the start of the curriculum, students discussed what symmetry is and the ways in which integers are and are not symmetric. Symmetry references were made throughout the remainder of the curriculum as well as being incorporated into the folding mechanic of the gameplay. Zero was emphasized as the midpoint between the positives and negatives as well as the origin.

Condition-Specific Computer Games

Students played computerized versions of the Leftovers game to begin experiencing a scaffold-fading transition (Pea, 2004) from manipulative representations of integers to symbolic notations. In the computerized games, students filled in numbers and number sentences to accompany the virtual manipulatives they saw on the screen. The games called specifically for children to translate concrete representations to symbolic representations and provided rapid feedback on each turn. Virtual manipulatives have been shown previously to positively impact integer learning (e.g., Bolyard & Moyer-Packenham, 2012). The transition to symbolic notation was further supported by paper-based activities in which students were asked to imagine their condition-specific perceptuo-motor activities to solve symbolic number sentences.

Condition Similarities

All conditions covered basic integer properties, conventions, and addition. All students were taught the following relational properties of integers: order, interval (equivalent spacing between consecutive integers), magnitude, and absolute amount. Importantly, all conditions learned about the additive inverse property, although only the folding condition was taught to think of it in terms of symmetry. All conditions were taught representational conventions such as how to write a negative number, were led in discussions of real-world examples of what negative numbers could stand for, and followed the same schedule of activities in the same order.

RESULTS AND MEASURES

Four types of integer knowledge were assessed: symmetry strategy choice, number line imagination speed/interference, generativity in new mathematical situations, and basic fluency.

Interrater reliability was assessed for all subjectively scored test questions using data from 30% of students. Agreement rates were > 85% on all measures ($M = 93\%$).

TABLE 4
Symbolic Fluency Accuracy at Pretest

<i>Condition</i>	<i>M (SD)</i>
Jumping	.55 (.34)
Stacking	.54 (.33)
Folding	.49 (.34)

Result 0. Prior-Knowledge Covariate: Integer Symbolic Fluency

Prior knowledge of integer symbolic fluency was assessed at pretest. The measure comprised 16 integer addition problems (e.g., $-4 + 9 =$) and 11 integer magnitude comparisons (Fill in $>$, $<$, or $=$, e.g., $0 \square -8$). Negative numbers had not yet been taught in school, but some children had learned about them elsewhere. Scores on the measure are shown in Table 4. Pretest scores were used before the start of the study to approximately equate prior knowledge across the instructional groups. The pretest was used as a covariate in all statistical analyses to control for effects of prior knowledge not accounted for by group stratification.

Result 1. Problem-Solving Strategy Differs Between Conditions and Predicts Performance

Strategy choice was measured to test whether symmetry framing led students to develop a symmetry orientation toward integer problem solving. Five students did not complete these measures due to absence, resulting in 25, 23, and 21 students measured in the folding, stacking, and jumping conditions, respectively.

Measures

Strategy was assessed during individual videotaped interview observations of integer problem solving. During each interview, a researcher sat next to the student at a table to administer and observe the task. For each problem, the researcher provided the student a sheet of paper with the problem written on it (Appendix A) and gave instructions verbally. Responses were coded for use of symmetry strategy and response accuracy. Three measures were included, as described below.

Tick-mark placement. Students were presented an empty number line with two tick marks drawn, one in the middle with the label 0 and one elsewhere with a corresponding integer label. Students were asked to draw a tick mark for a specified third integer. The instructions were, “Show me where the [X] belongs on the number line by drawing a tick mark where it goes. Try to be as exact as you can.” Three of these problems comprised the measure.

Symmetry-oriented strategies were identified. For example, when students were asked to place $+4$ given 0 and -4 on the number line, one symmetry-oriented strategy observed was to measure the distance from 0 to -4 with a thumb and index finger and use that distance to find the $+4$. One non-symmetry-oriented strategy observed was to start at 0 and draw four tick marks to the

right, labeling the fourth as +4. Students received credit on the measure if they used a symmetry-oriented strategy on any of the three questions. Accuracy was measured as absolute error in mm from the correct tick placement; it was averaged across the three problems.

Multiaddend addition. The problem was $1 + -3 + -2 + 5 + -1 = \dots$. Students were instructed, "Solve the addition problem on the page, but remember, you can draw or write stuff down to help you." The problem required students to extend their knowledge from the two-addend problems practiced during instruction.

Strategy was coded as 1 or 0 for being symmetry oriented or not, respectively. Symmetry-oriented strategies included reorganizing the problem to find sets of positives and negatives that make zero (+1 and -1; +5 and the sum of -3 and -2) and reorganizing the problem by lumping the positives together and the negatives together, which makes the balance between the positives and negatives apparent. A common non-symmetry-oriented strategy was to add the numbers sequentially from left to right. Accuracy was coded as a 1 or 0 for correct or incorrect response, respectively.

Number line labels. Students labeled an empty number line with the numbers -4 through 4. The numbers were shown clustered out of order in a box on the page, but students were told, "The numbers here are -4, -3, -2, -1, 0, 1, 2, 3, and 4. Fill in the number line with all these numbers." Symmetry strategy was coded as 1 or 0. Symmetry-oriented strategies included placing the zero first and alternating between placing the negative and positive versions of each remaining number. An example of a non-symmetry-oriented strategy was to start with -4 and work from left to right, ending with +4. Accuracy was determined by the order of the numbers. Other than the number line labels measure, students had never engaged in these tasks before.

Composite Score

The composite symmetry strategy score was the sum of the three measures, for a maximum possible score of 3.

Results

The folding group had higher symmetry strategy orientation scores than the other groups (Figure 7). An ANCOVA compared the composite symmetry strategy score for the three levels of condition (folding, stacking, jumping) while controlling for pretest symbolic fluency. The effect of condition was significant ($F(2, 65) = 5.07, p = .009$), such that the folding group used symmetry-oriented strategies more than the stacking and jumping groups (folding vs. stacking: $F(1, 65) = 5.54, p = .022, d = .710$; folding vs. jumping: $F(1, 65) = 9.01, p = .004, d = .932$). Post hoc comparison showed that the stacking and jumping groups did not differ, $F(1, 65) = 0.5, p = .483, d = .223$. The pretest covariate was not significant, $F(1, 65) = 0.44, p = .51$.

A second analysis showed that choosing a symmetry-oriented strategy was associated with better accuracy (Table 5). A MANCOVA was run with two dependent variables (accuracy on multiaddend addition and accuracy on tick mark placement) and two independent variables of condition (folding, stacking, jumping) and symmetry strategy score (a continuous measure), after controlling for pretest symbolic fluency. Symmetry strategy score significantly predicted

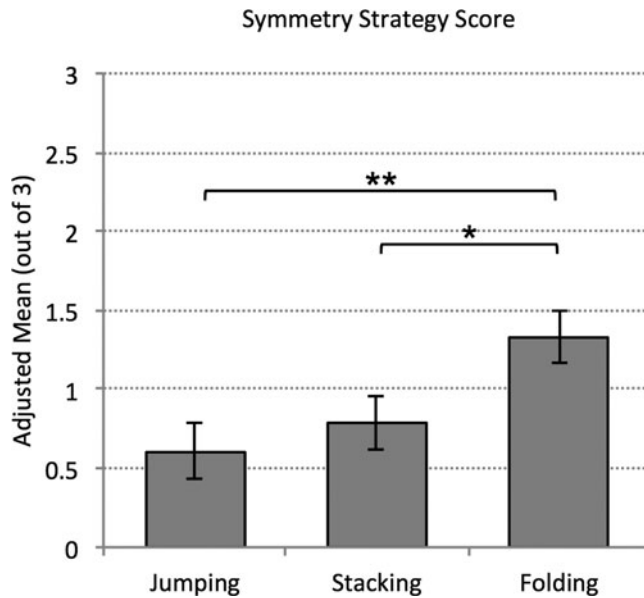


FIGURE 7 Symmetry Strategy Score, adjusted to control for pretest symbolic fluency. The folding condition scored higher on average than the stacking condition and the jumping condition. Error bars = ± 1 s.e. from the mean. * $p < .05$; ** $p < .01$.

accuracy (Wilks' Lambda $F(6, 114) = 4.72, p < .001$), such that greater symmetry orientation was associated with higher accuracy on both measures. The conditions were not different from each other overall ($F(4, 114) = .76, p = .553$). The advantage of symmetry strategy on accuracy did not interact with condition ($F(10, 114) = .74, p = .683$). The main effect of the pretest covariate was significant ($F(2, 56) = 3.26, p = .046$). (A log-linear analysis of the data yields similar results.) The number line labels measure was not included in the analysis as nearly all children produced an accurate number line.

TABLE 5
Accuracy on Symmetry Strategy Measures, by Symmetry Strategy Use Score

Measure	Symmetry Strategy Score			
	0 ($n = 25$)	1 ($n = 26$)	2 ($n = 16$)	3 ($n = 2$)
Tick-mark Placement (M(SD) absolute error in mm)	8.71 (3.92)	4.44 (3.50)	5.00 (3.92)	4.56 (2.40)
Multi-addend Addition (% correct students)	27%	48%	77%	116% ^a

Note. Means and SDs adjusted for pretest symbolic fluency score.

^aValues > 1 due to covariate adjustment.

Overall, the strategy data show that (a) some students took a symmetry-oriented approach to integer tasks, even tasks they have never done before; (b) this occurred most often for students in the folding condition; and (c) this orientation was beneficial for performance.

Result 2. Interference: Folding Group Slower to Imagine Nonsymmetric Number Lines

An interference measure tested the impact of the curricula on fast integer processing, as a way to gain insight into students' mental representations. The data are response times on the Zero Zone computer task, a set of number line imagination problems.

Rationale

Speeded interference measures help identify components of mental representations. During speeded tasks, subjects try to think about task-relevant features of the presented stimuli and try to ignore irrelevant features. If irrelevant features intrude on processing, however, people's response times are impacted in predictable ways. Researchers interpret predictable speed changes in interference tasks as indicating that an irrelevant feature is automatically being brought to mind as a part of the subject's mental representation of the task domain.

As a classic example, adults can very quickly determine which symbol is physically bigger, 3 versus 5. However, when the stimuli are changed to 3 versus 5, such that the 3 is physically bigger but numerically smaller, then response times slow down (e.g., Cohen Kadosh & Henik, 2006; Pinel et al., 2004; Tzelgov et al., 1992). The numerical magnitude of the numbers interferes with subjects' ability to judge physical size. Researchers use the interference as evidence that people automatically process magnitude when they see number symbols, and that the magnitude perceptuo-motor system is incorporated in subjects' representation of numbers. This interference effect increases from childhood to adulthood (Girelli, Lucangeli, & Butterworth, 2000; Rubinsten, Henik, Berger, & Shahar-Shalev, 2002), presumably as mathematical and numerical expertise mature, and its development corresponds to increased activity in areas of the brain that help people control and inhibit conflicting information (Wood, Ischebeck, Koppelstaetter, Gotwald, & Kaufmann, 2009).

In the current study, if we see response-time interference on problems where symmetry is irrelevant, it would suggest that symmetry is incorporated into students' mental representations of the problems.

Measure

Students completed the activity individually on a laptop computer. On each trial a number line appeared on the screen with only the endpoints labeled (Figure 4). A solid green box occluded part of the line. Students followed the instructions, "If you fill in the other numbers, does the ZERO GO IN THE GREEN ZONE?" They answered yes or no by pressing labeled buttons on the keyboard. The factor of interest was symmetry: Symmetric trials were those on which the endpoints were symmetric across (a hidden) zero (e.g., endpoints -5 and 5). Nonsymmetric trials

were those on which the endpoints were not symmetric across zero (e.g., endpoints -4 and 8). On nonsymmetric trials, symmetry is irrelevant and even detrimental because seeing symmetry in the presented number lines is misleading and requires the responder to spend an extra moment to rethink the problem.

There were three unique symmetric number lines (endpoints -5 and 5 , -6 and 6 , and -7 and 7) and six unique nonsymmetric number lines of interval size 12 (e.g., endpoint -5 and 7). Each number line was shown three times, once with the green box occluding a region to the left of 0 , once with the box occluding 0 , and once with the box occluding a region to the right of 0 , for a total of 27 task trials (9 symmetric, 18 nonsymmetric). Accuracy and response times were recorded. An additional four practice trials and one dummy trial were included at the start of the task but eliminated from the data set before analysis.

Thirteen students did not do the activity due to absence or scheduling constraints. An additional student was eliminated prior to analysis for answering only the first six trials. This left 22 , 20 , and 19 participants in the folding, stacking, and jumping conditions, respectively. Trials were excluded from the analysis if not answered within the 15 -s allotment ($M = 0.36$ affected trials per student, $SD = 1.28$, range: $0-9$). Trials were also removed if the response time was less than 100 ms ($M = 0.77$ trials per student, $SD = 1.94$, range: $0-9$). For each student, trials with outlier response times also were eliminated, defined separately for each student as response times greater than 2.5 standard deviations away from the student's mean ($M = 0.85$ trials per student, $SD = 0.60$, range: $0-2$). One trial was eliminated due to a computer error.

Results

Average accuracy on the task was 65% correct ($SD = 17\%$). It did not differ between groups. Average mean response time was 2554 ms ($SD = 1330$). It did not differ between groups.

A repeated-measures ANCOVA on response time was run with a 3-level between-subjects factor of group (folding, stacking, jumping) and a 2-level within-subjects factor of problem type (symmetric, nonsymmetric), controlling for pretest symbolic fluency. Accurate and inaccurate trials were included in the analysis to increase power. The main effect of group was not significant ($F(2, 57) = .83, p = .443$). The main effect of problem type was significant ($F(1, 57) = 9.44, p = .003$), such that nonsymmetric problems were slower than symmetric problems. Crucially, there was a significant interaction between group and problem type ($F(2, 57) = 4.94, p = .010$), such that the folding group slowed down more for nonsymmetric problems compared to symmetric problems than the stacking (marginal difference; $p = .071$) and jumping ($p = .003$) groups (see Figure 8).

A post hoc analysis confirmed that response time interference patterns did not differ significantly between accurate and inaccurate trials. A linear mixed model was run predicting response times by group, problem type, and accuracy (accurate vs. inaccurate trials), controlling for pretest symbolic fluency. Similar to the above analysis, results revealed a differential interference effect by group (group \times problemtype: $F(2, 164) = 5.98, p = .003$). There was no significant difference in the interference effect between accurate and inaccurate trials (group \times problemtype \times accuracy: $F(2, 163) = 2.20, p = .114$).

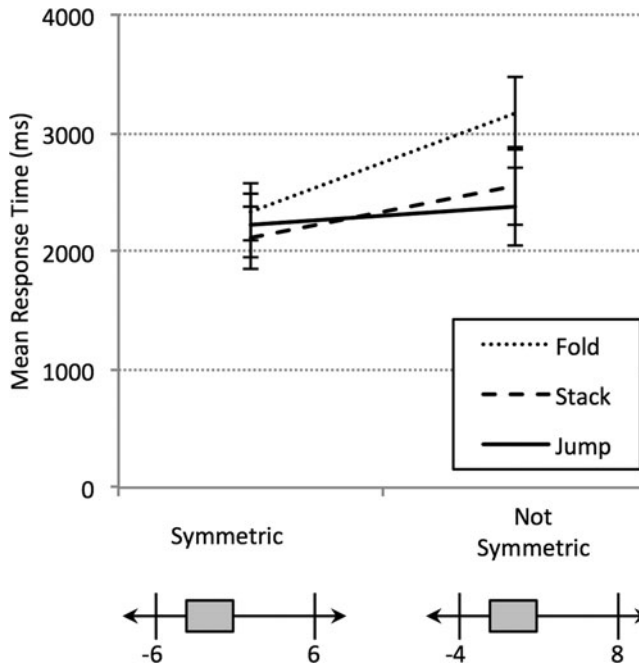


FIGURE 8 Zero Zone mean response times by group, for symmetric and nonsymmetric problems, adjusted to control for pretest symbolic fluency. The folding condition showed a marginally greater difference between symmetric and nonsymmetric problems than the stacking group ($p = .071$) and significantly greater difference than the jumping group ($p = .003$). Error bars = ± 1 s.e. from the mean.

Overall, the results show that students in the folding group were relatively slower than students in the other groups on nonsymmetric problems compared to symmetric problems. The findings indicate symmetry interference: When the symmetry structure of integers is built into their mental models, people interpret integer situations in terms of symmetry even when doing so is not useful or, in fact, is detrimental to their performance.

Result 3. Generativity: Folding Condition Most Accurate on Measures Requiring Extension of Learning

The generativity composite tested the influence of the curricula on students' performance on unfamiliar mathematical situations and topics beyond the basic integers.

Measures

There were four paper-and-pencil measures included in the generativity score (see Appendix B for measures).

Algebra-readiness. Students filled in the blanks in number sentences (e.g., $-5 + [] = 0$; $1 + -4 = [] + -2$). This topic had not yet been taught in the students' regular classes. The measure included seven problems and was scored as percent correct.

Negative fractions. A number line was shown labeled with three positive rational numbers ($\frac{3}{7}$, $1\frac{1}{2}$, and 2) and zero (Table 2). The negative versions of the numbers were shown elsewhere on the page, and students were asked, "Where do these numbers go on the number line? For each, draw a tick mark and write the number." Fractions are a notoriously difficult topic (Ni & Zhou, 2005). The students were familiar with fractions from their classroom curricula but had not yet mastered them. They had not been taught negative fractions before. Students received 1/2 point for putting the numbers in the correct order and 1/2 point for giving appropriate interval sizes between the fractions.

Even-teams word problem. This measure emphasized balance. The problem was, "Two teams play tug of war. The negative red team has three players. The positive blue team has two players. The blue team says, 'This is not fair!' The blue adds two more players. How many players should the red team add to make it fair?" One point was awarded for a correct response.

Spatial-positions word problem. This measure involved accurate spatial models. The problem was, "A negative red player and a positive blue player are playing catch on a basketball court. The red player is six feet away from the center of the court. The red player throws his ball ten feet to the blue player. How many feet is the blue player from the center of the court?" One point was awarded for a correct response.

Both word problems could likely be answered correctly without knowledge of negative numbers, but we posit that focusing on symmetry should facilitate generating mental models of these unfamiliar problems.

Composite Score

The generativity score was the sum of the four submeasures, for a maximum possible score of 4.

Results

An ANCOVA compared conditions on generativity while controlling for pretest scores. The main effect of condition was significant ($F(2, 70) = 5.89, p = .004$), such that the folding condition scored higher than the stacking and jumping conditions (folding vs. stacking: $F(1, 70) = 4.14, p = .046, d = .587$; folding vs. jumping: $F(1, 70) = 11.57, p < .001, d = 1.029$). Post hoc comparison showed that the stacking and jumping conditions were not different from each other ($F(1, 70) = 2.13, p = .149, d = .441$). The pretest covariate was significant ($F(1, 70) = 33.55, p < .001$) (see Figure 9 and Table 6).

A secondary analysis asked whether performance was related to the degree of interference shown on the Zero Zone measure. We found that across all conditions, a very high score on the

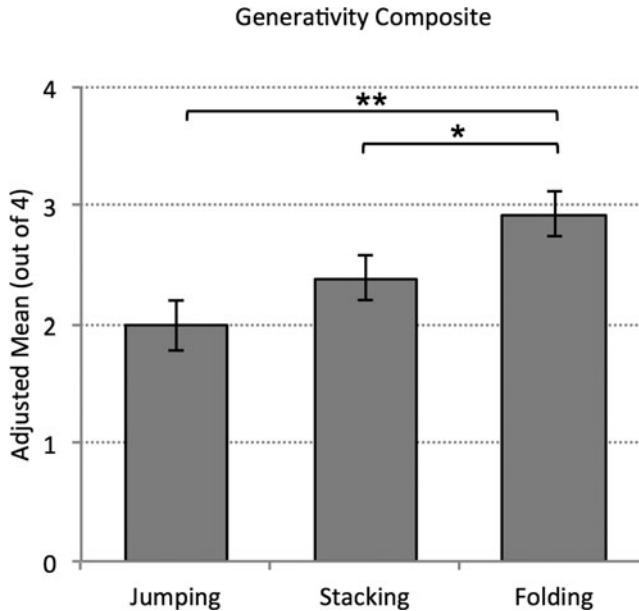


FIGURE 9 Generativity scores for each instructional condition, adjusted to control for pretest symbolic fluency. The folding condition scored higher on average than the stacking condition and the jumping condition. Error bars = ± 1 s.e. from the mean. * $p < .05$; ** $p < .01$.

generativity measure corresponded to greater interference. Students were categorized as high-scoring on generativity if their score was greater than 3 ($n = 17$) and not high-scoring otherwise ($n = 44$). A repeated-measures ANCOVA on Zero Zone response time was run with a 2-level between-subjects factor of generativity score (high, lower) and a 2-level within-subjects factor of Zero Zone problem type (symmetric, nonsymmetric), controlling for pretest symbolic fluency. Symmetric problems were faster overall than nonsymmetric problems ($F(1, 58) = 20.71, p < .001$). High generativity scorers were no different from lower scorers in overall response time, $F(1, 58) = 1.24, p = .271$. However, the interaction between problem type and generativity score was significant ($F(1, 58) = 11.75, p = .001$), such that students who did exceptionally well on

TABLE 6
Adjusted Mean (SD) Accuracy on Generativity Measures

Measure	Folding	Stacking	Jumping
Algebra readiness	0.72(.23)	0.64(.22)	0.59(.22)
Negative fractions	0.69(.41)	0.57(.39)	0.38(.40)
Even-teams problem	0.93(.40)	0.69(.38)	0.68(.40)
Spatial-positions problem	0.60(.45)	0.59(.43)	0.35(.44)
Total	2.93(.96)	2.39(.96)	1.99(.96)

Note. Means and SDs adjusted for pretest symbolic fluency score.

TABLE 7
Adjusted Mean (*SD*) Accuracy on Basic Fluency Measures at Posttest

<i>Measure</i>	<i>Folding</i>	<i>Stacking</i>	<i>Jumping</i>
Symbolic fluency	.81(.20)	.69(.20)	.74(.20)
Properties	.64(.19)	.60(.19)	.59(.19)

Note. Means and *SDs* adjusted for pretest symbolic fluency score.

the generativity task slowed down more for nonsymmetric problems than other students did. The pretest covariate was not significant, $F(1, 58) = .354, p = .62$.

In sum, the folding condition did better than the stacking and jumping conditions on measures requiring them to be generative when faced with unfamiliar mathematical topics that capitalize on balancing and symmetry. Furthermore, across all groups, students with high generativity scores showed more interference on the Zero Zone measure, suggesting that the strength of one's symmetry representation is related to one's ability to accurately interpret novel but relevant mathematical situations.

Result 4. Basic Fluency: No Condition Differences on Integer Symbolic Fluency or Knowledge of Properties

The two basic fluency measures tested fluency with integer symbolic operations and knowledge of integer properties, respectively, which all three conditions were designed to teach equally well.

Symbolic Fluency Measure

The measure included 22 addition problems and 16 number comparison problems. Addition included negative problems ($-1 + -4$), mixed problems ($6 + -2$), symmetric problems ($7 + -7$), problems of adding with zero ($0 + -8$), and subtraction with a negative answer ($2 - 5$). Number comparisons included negative, mixed, symmetric, and compare-with-zero problems. Four comparison and four addition problems with only positive numbers were also included but were eliminated before analysis due to ceiling effects. Overall, this measure was analogous to the symbolic fluency pretest, but included more problems.

Accuracy was the average of integer addition percent correct and number comparison percent correct (Table 7).

Symbolic Fluency Results

A repeated-measures ANOVA was run on symbolic fluency accuracy with the within-subjects factor of time (pretest, posttest) and the between-subjects factor of condition (folding, stacking, jumping). There was significant improvement from pretest to posttest ($F(1, 71) = 56.47, p < .001$). The conditions were not different from each other overall ($F(2, 71) = 0.11, p = .893$), and the interaction between time and condition was not significant ($F(2, 71) = 2.04, p = .138$).

Integer Properties Measure

Eight paper-and-pencil problems tested knowledge of relational properties that were taught to all groups: order, interval, magnitude, polarity, and understanding of addition. Many of the problems were based on Saxe and colleagues (2010). Exact problems are shown in Appendix C.

Of the eight problems, five were incorrect or incomplete number lines which students were asked to complete or to assess for accuracy. For example, one number line included tick marks at the positions of -4 , -3 , -1 , 0 , 1 , 2 , and 4 , but only 0 was labeled. Students were asked to label the other tick marks. This question assessed whether students would put negatives on the appropriate side of zero and whether they would notice the interval between the tick marks. Another question simply presented a number line with seven tick marks evenly spaced along the line and asked students to label 0 , 1 , and -1 . This tested whether students knew that the negatives belong to the left of zero on the number line.

Two nonnumber line problems asked students to write integers in order of magnitude. One showed the following numbers in a cluster on the page: -1 , 2 , -3 , 4 , and -5 . Students were asked to write the numbers in order from least to greatest. A correct response indicated knowledge of order, magnitude, and polarity. The other problem asked students to write two numbers more than -4 and two numbers less than -2 . A final question probed students' understanding of addition by asking, "Ethan adds a number to the number 8. Can his answer be smaller than 8?" Accuracy was scored as percent correct out of eight (Table 7).

Properties Results

An ANCOVA was run on accuracy with the between-subjects factor of condition (folding, stacking, jumping) while controlling for pretest symbolic fluency. Condition did not predict properties score, $F(2, 70) = 0.32$, $p = .730$. The pretest covariate was significant, $F(1, 70) = 23.36$, $p < .001$. The lack of significant condition differences on either basic fluency measure suggests there were no significant differences in instructional quality for basic milestones of integer knowledge.

DISCUSSION

Review of Results

We began by asking how perceptuo-motor systems can be used to support mathematics instruction, and we posited that by combining ideas on the genesis of mathematical thought with ideas about mathematics instruction, we could design effective learning experiences. We devised three instructional conditions to test if we could improve students' understanding of the integers by emphasizing the symmetry about zero, which we believe supports the additive inverse property of the integers. Results showed that the symmetry-focused folding condition performed predictably differently from two control conditions on several measures.

The most notable result, in our view, was that symmetry-based curriculum helped students solve problems beyond the scope of the instruction. Students in the folding condition outperformed

their jumping and stacking peers on the generativity measure of algebra-readiness problems, negative fractions problems, and unfamiliar word problems. Neither algebra-readiness concepts nor negative fractions had yet been introduced in the students' classes. The results are consistent with mental model theory (Johnson-Laird, 1983), which suggests that a mental model of integers that closely mimics the axiomatic structure of integers can enable the holder to think generatively about new related topics. The folding condition may have given students mental models that strongly implemented the additive inverse structure of integers, which in turn helped students generate well-structured models to solve the generativity problems. The result suggests that symmetry could be a valuable addition to current integers curricula in schools.

The study further showed that on integer problem solving tasks, students in the folding condition chose more symmetry-oriented strategies than students in the stacking and jumping conditions. Some stacking and jumping students chose symmetry strategies without explicit instruction, suggesting a degree of spontaneous symmetry noticing. However, rates were low; less than half of stacking and jumping students used symmetry strategies on any of the three strategy measures, compared to almost 90% of students in the folding condition.

The speeded interference task, the Zero Zone, provided further evidence regarding students' incorporation of symmetry. Students in the folding condition showed greater interference for nonsymmetric compared to symmetric trials than control students. Our interpretation is that folding students capitalized on symmetry to make sense of problems and did so to such a degree that it acquired some degree of automaticity, even in situations where it was not helpful (as on nonsymmetric Zero Zone trials). We also found that across all conditions, students who did the best on generativity problems showed the greatest symmetry interference effects.

It is important to note that the interference effect does not imply that symmetry in integer mental models is maladaptive. Rather, it reflects that the structures in our mental models affect the way we perceive and act on information. If a perceptuo-motor capacity supports a stable or standard structure—such as the additive inverse of integers—it generally helps us navigate the world. However, it can hinder our performance in odd situations in which the capacity is irrelevant or at-odds with the goal of the task. For example, being able to read quickly and automatically is ordinarily good, but it is detrimental in a Stroop task when asked to name the ink color of color words, like “WHITE” (printed in black ink) (Stroop, 1935). Fortunately, fine-grained response time tasks like the Stroop task or the Zero Zone are not common in school or life. Thus, the drawbacks of the interference effect are small compared to, say, the benefits of being able to generalize one's knowledge to new school topics.

The final assessment was a control measure crucial from a methodological standpoint. We found that all groups improved similarly on the symbolic integer fluency measure, which targets types of problems that all groups had been taught and practiced. They also performed similarly on probes of their knowledge of nonsymmetry integer properties. These results indicate that all instructional conditions received equally effective instruction, and therefore, differences found on other measures are not due to poor instruction, differences in engagement, or even differences in basic integer property learning. Rather, the results on noncontrol measures stem from differences in integrated symmetry knowledge specifically.

Broadly, the study demonstrated that even for abstract numbers, people can learn via experiences with concrete manipulatives so long as the experiences make apparent the relational structure that defines the numbers. Symmetry emphasis in our curriculum provided structural constraints on manipulatives use that led students to develop an effective and generative mental model of integers.

Designing Instruction for Structure

In this section, we discuss the process of designing instruction for numerical structure based on perceptuo-motor capacities and experiences. We demonstrated in the research that one way to approach mathematics instruction is to focus on exposing the perceptuo-motor computations that underlie important mathematical structure. The practical question is, how? There are likely multiple effective ways. We chose a method of guiding students to perceive and internalize the mathematical structure displayed in concrete manipulatives. Here we describe our method and the corresponding instructional techniques we employed to usher students through the process.

First, as students use manipulatives, they must simultaneously notice the new and old numerical relations portrayed therein and realize that the structure in the manipulatives reflects formal numerical structure. To encourage this in the current study, students used the manipulatives in specific ways intended to make the structure stand out (e.g., symmetry via folding the number line in half); and instructors held focused class discussions to point out relations to notice in the manipulatives.

Second, students must establish connections between the manipulatives used and the corresponding symbolic representations of the numbers. To this end, students used multiple representations of integers simultaneously in game play, including number lines, blocks, playing cards with symbolic and pictorial representations of integers, and written numerical expressions.

Third, the eventual goal is for students to internalize mathematical structure. In preparation, students should become comfortable with the actions performed on the manipulatives, as well as the numerical structure portrayed in the manipulatives. To set students up for internalization, students practiced their manipulative mechanics many times through concrete and virtual games, such that students could do the mechanics easily and accurately.

Fourth, as internalization occurs it may take the form of imagery—imagining the manipulatives—which should both simulate the manipulative actions and emphasize the important structural relations of the manipulatives. Students should know that this imagery is applicable to symbolic problems, concrete problems, and new relevant mathematical situations. A typical use may be to read or hear a math problem, figure out how to simulate it with imagery, and read off the result from one's mental image. To encourage imagery, our curriculum included a set of prediction activities that specifically asked students to imagine their game mechanics.

Fifth, eventually the simulative aspects of the imagery should fade, so that during problem solving, students' mental models of problem situations include the relevant integer structure independent of explicit simulative imagery. Importantly, physical manipulation or simulation is not required to recruit the functional computation of symmetry; eventually this structure becomes enforced without these actions. Traces of the simulation or motion likely remain, however. For instance, adults' neural signals during mental calculation with natural numbers shows residual traces of finger counting, even during simple multiplication fact retrieval on which subjects presumably are not physically or mentally counting on their fingers (Butterworth, 2006).

Notably, throughout the entire process of noticing and internalizing, students should continually connect and integrate new numerical structures with known numerical structures. We saw evidence in pilot work that children need targeted activities to integrate symmetry with other numerical properties such as magnitude. Earlier versions of the folding condition did not include activities explicitly emphasizing integration. Students' resulting mistakes suggested they had failed to integrate symmetry with magnitude (i.e., divided number line model, Peled, Mukhopadhyay, & Resnick, 1989). For example, they confused the left-right polarity of the negative

and positive integers. In the current study, curriculum included several activities geared toward integrating symmetry with other integer structures. Integration was encouraged in games by concurrently highlighting multiple relational structures and in short activities that asked students to think about several number relations together. For instance, one activity asked the folding group to generate a bar graph representation of the integers that shows magnitude, order, and symmetry simultaneously.

Related Approaches

Our approach to designing hands-on materials differs from related approaches and theories of embodied cognition. Several embodiment theories emphasize abstraction from perceptuo-motor experiences. DiSessa (1993), for instance, has posited the centrality of phenomenological primitives, such as springiness, that undergird more sophisticated physics concepts. These primitives are based in intuitions stemming from interaction with the world and have been argued to lend meaning even to symbolic equations (Sherin, 2001). Lakoff and Núñez (2000), for example, have proposed that mathematical concepts are metaphorical abstractions over bodily experiences—the concept of sets stems from the experience of objects inside versus outside the body. In this view, hands-on materials might be designed with a focus on finding useful metaphors to couple with learning activities.

Our version of perceptuo-motor learning differs from these proposals in that it takes as its starting point a lower level of cognition than concepts or metaphors, namely perceptuo-motor systems in the brain. Rather than focusing on abstractions from experience, we focus on the brain systems that enable one to experience a phenomenon in the first place. In our approach, hands-on activities serve a specific purpose: to focus attention on mathematical structure to support recruitment of perceptuo-motor capacities. We do not suggest that using hands-on materials is the only way to serve this purpose nor that the hands-on nature of the materials was itself what led to learning (as in ideas of kinesthetic learning styles). Rather, we suggest that manipulatives can be designed and used in a way that supports attention to mathematical structure and recruitment of perceptuo-motor capacities, and this can improve learning.

Limitations

This work has several limitations. First, the strategy results (strategy choice on tick-mark placement, etc.) were imperfect. They showed that while symmetry strategies are associated with better accuracy and the folding group used symmetry strategies the most, the folding group did not perform more accurately than the other groups. To begin to explain this, we looked at variability among symmetry strategy users on tick-mark placement accuracy. For students identified as symmetry users (symmetry strategy score of 1 or greater), the variability in accuracy was 3.7 mm (standard deviation) for the folding condition, compared to 1.7 mm for stacking and 1.0 mm for jumping. This suggests that while the folding curriculum drove more students to use a symmetry strategy, they varied in how effectively they applied the strategy compared to spontaneous symmetry users. Students who spontaneously used symmetry in the other conditions may have previously adopted a mental model that strongly incorporated symmetry, while those in the folding condition were newly developing it through instruction.

Another limitation is that the instruction was designed to support addition concepts but not other operations, as addition is the most relevant to the additive inverse property of interest in this study. It is not clear how well the instruction could be adapted to support integer operations such as subtraction or multiplication. It may be that another perceptuo-motor system would be more appropriate for these concepts. Alternatively, not all mathematical concepts are necessarily rooted in perceptuo-motor capacities. It is possible that these concepts are best supported by operational rules and conceptual metaphors. This study examines the theory of perceptuo-motor support for mathematics learning with respect to a single mathematical property. It leaves untested the question of whether perceptuo-motor systems would support other kinds of mathematical concepts.

Third, our integers instruction purposely presented different conceptual rationales for integer addition in each condition. For example, the jumping curriculum emphasized addition as movement, while the stacking and folding curricula emphasized addition as both movement and quantity cancellation. One possibility is that our results reflect students' differential concepts of addition rather than integer structure. However, addition concepts alone cannot explain the overall pattern of results. For instance, they cannot readily explain condition differences on non-addition measures, such as the interference measure and fractions problem. The complete pattern of results is more consistent with the current interpretation that the conditions adopted different mental models of integer structure. Generally, we take integer addition and integer structure to be inextricably related, as the additive inverse property defines crucial aspects of integer structure by way of addition.

A fourth limitation is loss of data due to absences and failure to turn in permission forms. This likely limited the study's statistical power and effect sizes compared to a larger sample. Notably, there was more data loss in the jumping condition than the other conditions. For example, the paper-and-pencil posttest was taken by 26, 26, and 22 students in the folding, stacking, and jumping conditions, respectively. We do not know why the jumping condition lost more students than the other conditions, but we speculate that any resulting bias in the data was conservative with respect to our hypotheses. In our experience, it is often (but not always) the less studious children who are absent or fail to turn in permission forms. These children's hypothetical data likely would have brought down the jumping group's averages, exaggerating the differences between jumping and other groups.

A final concern is a question of worth. In prior research we found that adults represent integer symmetry presumably without being taught it directly in their elementary schooling (Tsang & Schwartz, 2009). Additionally, in the current study we found that some students in the control conditions adopted symmetry without being taught explicitly. This begs the question, if some children notice integer symmetry spontaneously and most adults adopt it eventually, why spend the effort to teach it in an introductory integers curriculum? Our generativity results indicate that symmetry emphasis confers some benefits to students on extension problems. Although untested, emphasizing symmetry structure early may help prepare students to learn more difficult math topics in the future. Griffin et al. (1994) showed that early efforts to strengthen children's natural number structures set the children on a positive learning trajectory that put them well beyond control students a year after the study, especially on topics more advanced than those covered in the original treatment. We speculate that a similar effect could take hold from strengthening integer structures like symmetry early on.

A related concern is that the methodological approach we employed is very resource and time intensive, and this may prevent the methodology from being adopted widely in the future. To

answer the genesis and instructional questions to design curriculum involves, at the very least, an in-depth semantic analysis of each mathematical concept to be taught, followed by hypotheses about relevant perceptuo-motor systems, which then need to be supported empirically and turned into a curriculum. This kind of work requires a commitment to tackle a single concept deeply. An alternative approach might be to apply domain-general learning mechanics such as spaced practice (for a review, see Cepeda, Pashler, Vul, Wixted, & Rohrer, 2006) or self-explanation (Chi, De Leeuw, Chiu, & LaVancher, 1994).

CONCLUSION

This research demonstrates one way to bring perceptuo-motor capacities to bear in the teaching and learning of mathematical structure. Building on previous research showing that the capacity to perceive symmetry supports adult integer cognition, we found that when instruction was designed to lead fourth graders to “see” the powerful symmetric structure of the integers, students incorporated symmetry into their mental models of integers and were more effective than control groups at making sense of unfamiliar but related mathematical topics. The primary goal of the instruction was not to construct a metaphor for negative numbers or for students to experience negativeness itself. Rather, the instruction used perceptuo-motor experiences to focus student attention on the structure of the integers, specifically the symmetry around zero, helping students recruit their natural symmetry capacities to incorporate the structure into their mental representations of integers.

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APPENDIX A: STRATEGY CHOICE ASSESSMENT ITEMS USED ON POSTTEST

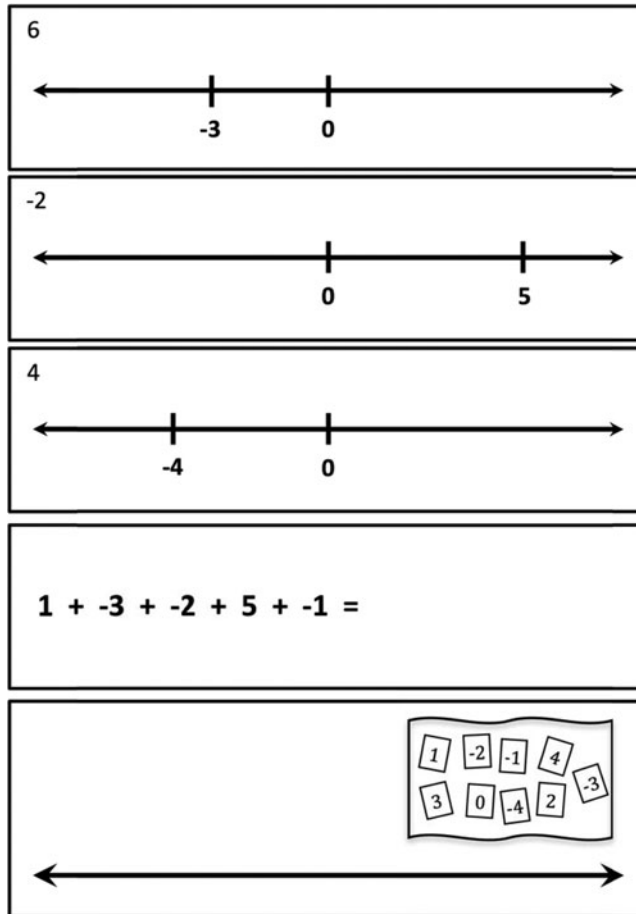


FIGURE A1 Strategy Choice Assessment Items. Each Problem was Presented on a Separate Sheet of Paper.

APPENDIX B: GENERATIVITY ASSESSMENT ITEMS USED ON POSTTEST

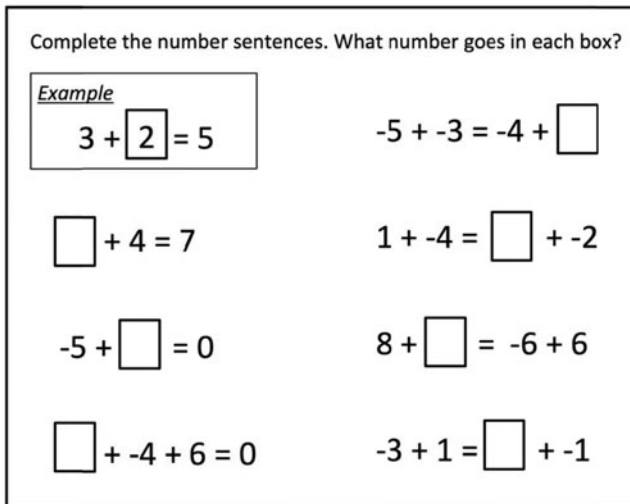
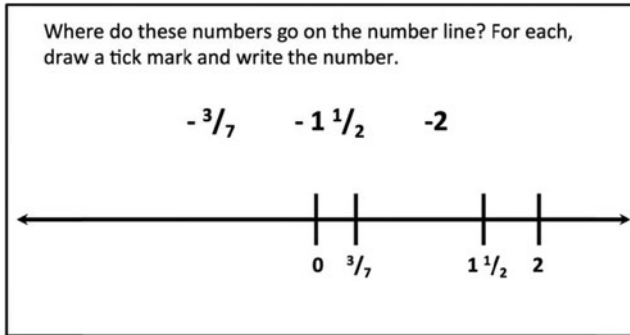


FIGURE B1 Generativity assessment, items 1–2 of 4.

Two teams play tug of war. The negative red team has three players. The positive blue team has two players. The blue team says, "This is not fair!" The blue adds two more players. **How many players should the red team add to make it fair?**

What is the answer?
Explain your answer here:

A negative red player and a positive blue player are playing catch on a basketball court. The red player is six feet away from the center of the court. The red player throws his ball ten feet to the blue player. **How many feet is the blue player from the center of the court?**

What is the answer?
Explain your answer here:

FIGURE B2 Generativity assessment, items 3–4 of 4.

APPENDIX C: BASIC FLUENCY ASSESSMENT ITEMS USED ON POSTTEST: SYMBOLIC FLUENCY AND INTEGER PROPERTIES

Solve the following problems:			
$9 + 4 =$	$5 + 8 =$	$7 - 4 =$	$11 - 3 =$
$-5 + 2 =$	$1 + -6 =$	$6 + -2 =$	$-1 + 5 =$
$-2 + 4 =$	$4 + -1 =$	$3 + -5 =$	$-9 + 1 =$
$-4 + 4 =$	$-18 + 18 =$	$7 + -7 =$	$0 + -16 =$
$-3 + -6 =$	$-1 + -4 =$	$-5 + -2 =$	$2 - 5 =$
$3 - 4 =$	$1 - 8 =$	$0 + -7 =$	$-5 + 0 =$
$-15 + 13 =$	$-18 + 21 =$		

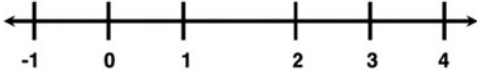
FIGURE C1 Symbolic Fluency assessment items: Arithmetic.

Compare. Write $<$, $>$, or $=$ for each

EXAMPLE: $2 < 7$

$3 < 5$	$12 < 10$	$4 < 14$	$7 < 6$	$-8 < -12$
$-6 < -2$	$-11 < -16$	$-5 < -21$	$4 < -7$	$-10 < 3$
$2 < -20$	$-8 < 9$	$-6 < 6$	$-14 < 0$	$17 < -17$
$-8 < 0$	$0 < -14$	$-5 < 5$	$0 < -4$	$2 < -2$

FIGURE C2 Symbolic Fluency assessment items: Number comparison.



Is this number line right or wrong?

RIGHT WRONG

If it's wrong, what's wrong? _____

Write these numbers on the number line

0 1 and -1


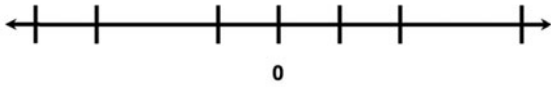


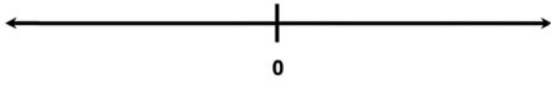
FIGURE C3 Integer Properties assessment, items 1–2 of 8.

Which numbers go with the tick marks? Write one number for each tick mark.



Where do these numbers go on the number line? Draw tick marks AND numbers.

-4 -2 -1 -3



Write the numbers in order from **least to greatest**

EXAMPLE:

4 7 1	11 3 0	-1 4 2 -3 -5
<u>1</u> <u>4</u> <u>7</u>	_____	_____

FIGURE C4 Integer Properties assessment, items 3–5 of 8.


<p>Write two numbers that are MORE than -4.</p> <p style="text-align: center;">_____</p>
<p>Write two numbers that are LESS than -2.</p> <p style="text-align: center;">_____</p>
<p>Where do these numbers go on the number line? Draw tick marks AND numbers.</p> <div style="text-align: center; margin: 10px 0;"> 101 -50 99 50 -101 0 50 -99 </div> <div style="text-align: center; margin: 10px 0;">  </div>
<p>Ethan adds a number to the number 8. Can his answer be smaller than 8?</p> <p style="text-align: center;"> <input type="checkbox"/> YES <input type="checkbox"/> NO </p> <p>Why or why not?</p> <p>_____</p> <p>_____</p>

FIGURE C5 Integer Properties assessment, items 6–8 of 8.

APPENDIX D: EXAMPLE CURRICULUM ACTIVITY: “WHAT IF THE NUMBER LINE LOOKED LIKE THIS?”

Context

This short worksheet lesson was given during Day 2 of instruction. It was intended to encourage students to notice some mathematical properties of the number line. The instructor explained the worksheet and students completed it individually. The children then gathered as a group and reviewed what they learned by responding to incorrect answers posed by the instructor.

The figure shows two side-by-side worksheets. Each worksheet begins with a horizontal number line from -4 to 4, with tick marks and arrows at both ends. Below the number line is the question: "What would the number line would look like if it we ...".

Left Panel:

- Question: "... tilted it?"
- Diagram: A square box containing a line segment tilted upwards from left to right, with arrows at both ends.
- Question: "... folded it?"
- Diagram: A square box containing a V-shaped line segment opening downwards, with arrows at both ends.
- Question: "... made it wavy?"
- Diagram: A square box containing a wavy line segment, with arrows at both ends.

Right Panel:

- Question: "... tilted it?"
- Diagram: A square box containing a line segment tilted upwards from left to right, with arrows at both ends.
- Question: "... folded it in half?"
- Diagram: A square box containing a V-shaped line segment opening downwards, with arrows at both ends.
- Question: "... bent it?"
- Diagram: A square box containing a line segment that is bent into a curve, with arrows at both ends.

FIGURE D1 “What if the number line looked like this?” worksheets for the jumping and stacking conditions (left panel) and folding condition (right panel).

Instructor Script

To administer the lesson, instructors followed a semi-structured script (Figure D2). The script and lesson were generally the same between conditions, but with specific differences corresponding to the numerical structures of emphasis in each condition.

- Worksheet (10min duration)
 - Hold up the worksheet and explain: “Great job! It’s time for a worksheet. You’re going to get a sheet like this. See this number line here, your job is to draw, what do you think it would look like if you ...
 - **F Group:** “Tilted it” [point to it] “bent it in the middle” [point to it] “folded it” [point to it]
 - **S & J Group:** “tilted it?” [point to it] “Folded it like this?” [point] “Made it wavy?” [point to it]
 - “Make sure you include ALL the tick marks. And make sure to label ALL the tick marks”
 - They can work on it at their desks.
- Discussion (5 min duration)
 - Come back to the floor. “Let’s go over it together. I’ll draw my answer, and you tell me if it’s right or not.”
 - Draw the following incorrect examples for **F Group**:
 - #1 interval and polarity – negs on wrong side AND one big space somewhere
 - #2 symmetry – neg spacing bigger. “It should be symmetric. So if you fold it all the way, the 3 and -3 should touch, and the 5 and -5 should touch.
 - #3 order within negs – backward neg order”
 - Draw the following incorrect examples **S Group and J Groups**:
 - #1 interval – one big space somewhere
 - #2 polarity – negs on wrong side
 - #3 order within negs – backward neg order
 - As you do each one, prompt for what’s wrong. Repeat it when they get it. Then correct your number line.
 - For second and third, make a big deal of showing that the previous criteria were met.

FIGURE D2 Lesson plan/script for “What if the number line looked like this?” lesson.