Symmetry in the Semantic Representation of Integers

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Abstract

The integers include more structure than the natural numbers; for example, they exhibit symmetry about zero. Do adult mental representations directly encode this increased structure? In two studies, adults completed a numerical bi-section task in which they were presented with two symbolic integers and were asked to report the digit at the midpoint of the interval. The reaction times demonstrated a "tuning curve" such that people were faster when the midpoint or end point of an interval was close to zero. The results suggest that the mental representation of negative numbers and zero has incorporated analog properties to represent the increased structure of the integers.

Keywords: analog representation; mathematics; bisection; symmetry; integers

Introduction

The positive integers, or natural numbers, have ready perceptual referents; for example, six divided by three can be materialized in the world as six cookies shared between three people. The structural properties of natural numbers (e.g., ordinality, cardinality, magnitude) can be gleaned from applying numbers to physical situations (Griffin, Case, & Siegler, 1994). This may help explain why people have an analog representation of natural numbers, as indicated by the symbolic distance effect (Moyer & Landauer, 1967) and more recent brain evidence (Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004).

In contrast, negative integers are abstract entities that do not map readily to tangible things; negative three cookies are hard to imagine, and a negative times a negative is typically handled symbolically in school curricula. Historically, negative numbers were considered fictitious even when effectively used in calculations (Schwarz, Kohn, & Resnick, 1993/1994). Schwarz, et al. (1993/1994) say, "It was only in the 19th century that negative numbers emerged as directed magnitudes (e.g., in the domain of electricity) and that the set of integers was axiomatically defined in such a way as to give negatives a symmetric status to that of positives."

The full set of integers includes greater quantitative structure than the natural numbers; for example, it includes zero as an identity; and it is possible for quantities to exhibit symmetry. How do people represent the increased structure of integers?

One possibility is that the new structure is carried symbolically by a set of manipulation rules and categories (e.g., if two digits are the same absolute amount and one has a negative sign, then they are symmetric). A recent study by Varma and Schwartz (2009) suggests that this is not the case for adults. The authors compared seventh graders to adults on a number comparison task. Seventh graders showed no symbolic distance effect when comparing a positive and negative number. The absence of this robust marker of analog representation suggests the children used a rule like "a positive is larger than a negative" to compare the numbers. On the other hand, adults showed an 'inverse' distance effect on these problems – they answered far comparisons slower than near comparisons – indicating that they had developed a semantic representation of integers.

Given that adults use a semantic representation to operate

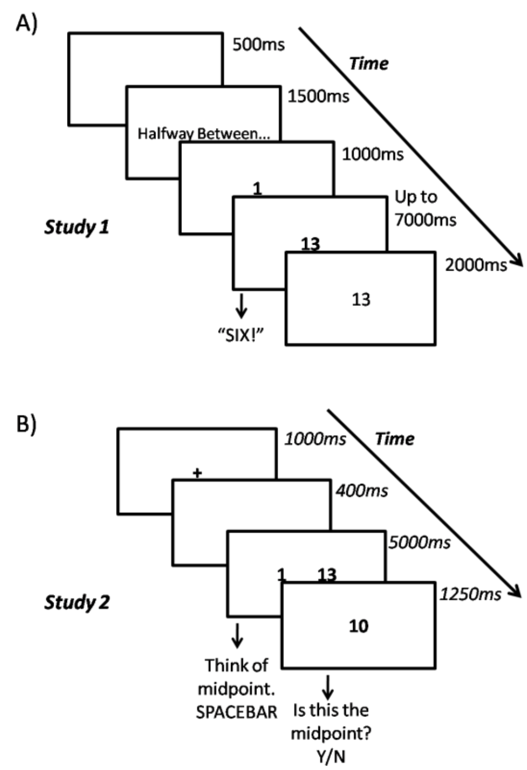

Figure 1: Protocol for bisection task. A) Study 1. B) Study 2.

with integers, it is possible that this representation directly encodes the increased structure of the integer system; for example, people may have developed an analog representation that encodes symmetry level as well as magnitude. This would be an interesting finding because people do not directly experience negative objects.

To examine this possibility, we focused on the two inherent structures of negative integers: (a) symmetry about zero and (b) identity for addition/subtraction, which also involves zero. Subjects completed two number bisection tasks (Figure 1). They saw two symbolic digits on a computer screen, and orally reported the midpoint (Study 1) or verified a third digit as the midpoint (Study 2). For example, given -4 and 2, they might report -1. If people solve these problems by applying a set of rules, then we would expect them to take about the same amount of time regardless of the proximity of the values to symmetry or identity. For example, -8 and 2 should take the same time as -4 and 6. Alternatively, if people have and use symmetry built into their representations of the integers, then we would expect them to solve problems closer to symmetric (-4 and 6) more quickly, because they can leverage relative symmetry about zero to find the midpoint. The same predictions apply for identity. By a rule-based account, -11 and 1 should require the same amount of time as -9 and 3. By a semantic account, -11 and 1 should be solved faster, because the 1 is closer to the identity of zero.

Both accounts can predict extremely fast reaction times for perfectly symmetric problems (-6 and 6) and perfect identity problems (0 and 12) (see Figure 2, top and bottom sections). The differential predictions involve how response times vary when the problem diverges from perfect symmetry and identity. The semantic account predicts a reaction time "tuning curve," indicative of analog processes. The symbolic account does not. By the semantic account, as intervals diverge from symmetry about zero people should slow down (e.g. -3 and 9 is slower than -5 and 4). As the interval moves toward the identity people should speed up again (e.g. -1 and 11 is faster than -3 and 9). Moreover, people should only exhibit symmetry effects for mixed pairs, in other words intervals bounded by a positive and negative integer.

Figure 2, top and middle, shows the Symmetry-to-Anchored Scale (STAS), which quantifies the degree to which a problem pair diverges from symmetry and identity. Identity pairs are termed "anchored" in the scale, because the 0 presumably anchors the problem solving.

Symmetric pairs (6 and -6) receive the value of -1, while anchored pairs (0 and 12) receive the value 1. STAS values for non-symmetric and non-anchored pairs were computed as follows:

$$STAS\ Value = -1 + \frac{4}{I} * |M| - 2 * \left\lfloor 2 * \frac{|M|}{I} \right\rfloor$$

Where:
I = size of the interval bounded by the stimulus pair
M = midpoint of the stimulus pair

For mixed pairs, STAS scores are based on the proportion of the interval's size represented by double the magnitude of the interval's midpoint. These proportions are shifted by a constant of -1 and stretched by a factor of 2 so that they span from -1 to 1. Since STAS scores are based on proportion of interval size rather than absolute numerical distance, the STAS allows intervals of different sizes to be collapsed onto the same -1 to 1 metric.

For non-mixed interval pairs (i.e. bounded by two positive or two negative integers), rather than quantifying the distance from symmetric and anchored, the STAS defines an interval's distance from anchored and "skip-counted" (e.g. the pair 5 and 15).

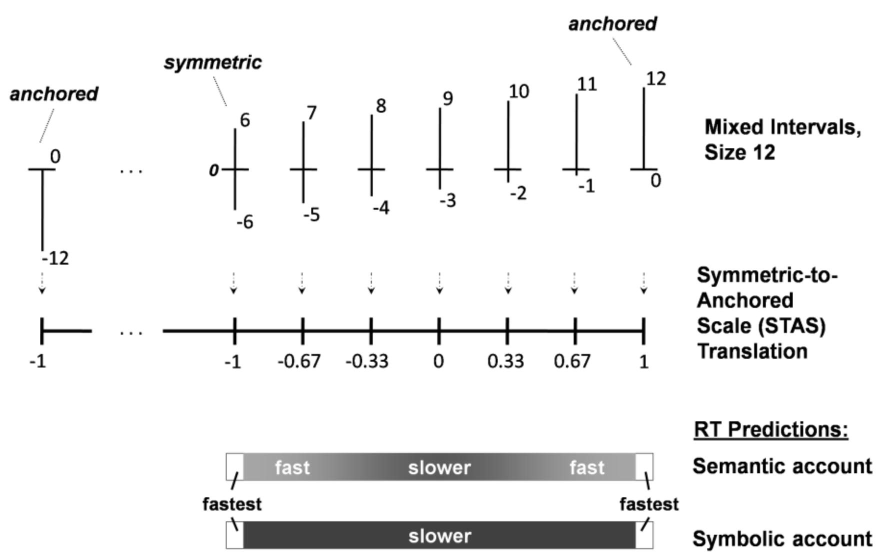

Figure 2: Symmetric-to-Anchored Scale (STAS) values and RT predictions for intervals of size 12 (those that span zero only). Top: The interval pairs. Symmetric pairs have midpoint 0. Anchored pairs have endpoint of 0. Middle: Corresponding STAS values. Zero denotes intervals maximally far from either symmetric or anchored. Bottom: Response time predictions according to the semantic and symbolic accounts of integer processing.

Study 1: Oral Response

In study 1, subjects completed an oral-response interval-bisection task including purely positive, purely negative, and mixed problems, as well as problems that involve zero as an interval endpoint ("anchored") or midpoint ("symmetric").

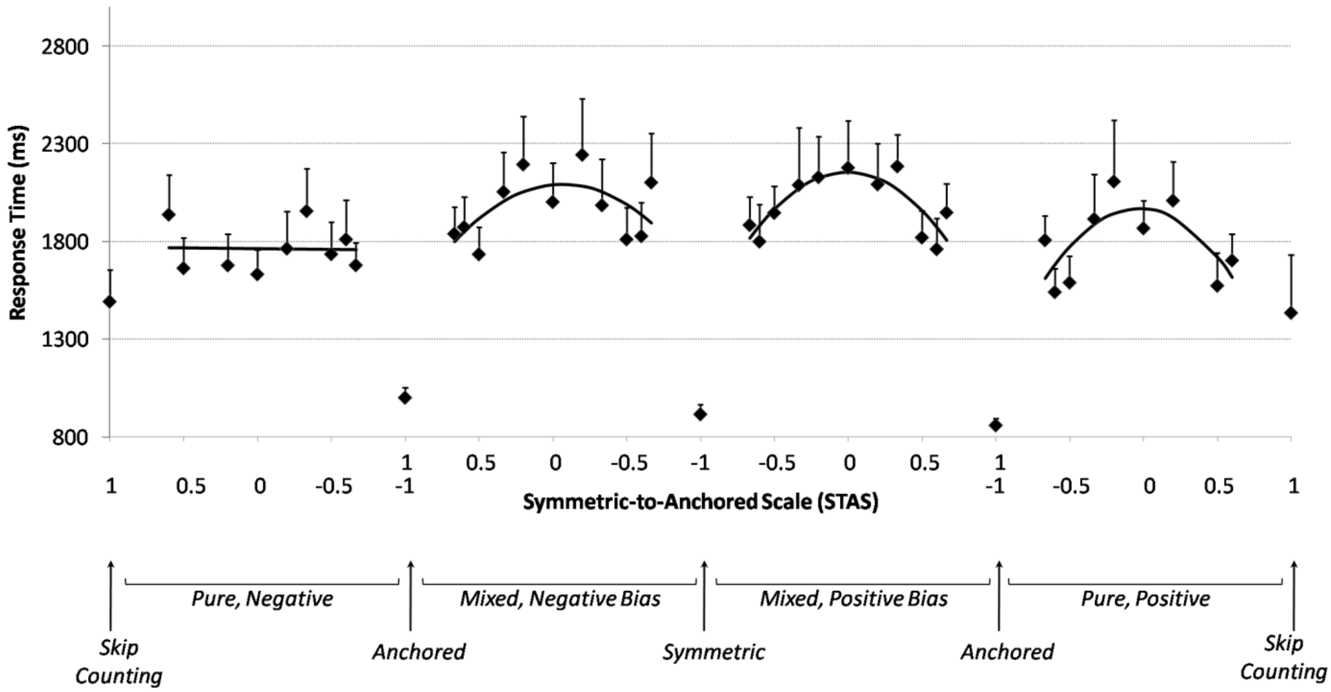

Figure 3: Study 1 response times by STAS value and problem type. Fitted lines come from multiple regressions on the group means at each response point. Lines were estimated for each problem type separately, and are curvilinear if the squared regressor in the model was significant at $p<0.15$. Error bars represent 1 s.e. between-subjects.

Methods

Participants Sixteen adults from a private university participated (8 female, mean age=30.1 years, *SD*=8.4).

Stimuli There were 88 unique pairs of integers. The integers ranged from -15 to 15. Each integer appeared between 8 and 16 times, with the numbers closer to 0 repeated more often than those farther from 0. Each pair defined an interval and there were four intervals sizes: 50 pairs of interval size 6; 46 pairs of size 8; 42 pairs of size 10; and, 38 pairs of size 12. Pairs were classified into five types: 48 pure positive (e.g. 2, 10), 48 pure negative (e.g. -2, -10), 56 mixed (e.g. 2, -10), 16 anchored (e.g. 10, 0), and 8 symmetric (e.g. -3, 3). Each pair was presented twice, flipping the order of the digits, for a total of 176 pairs.

Procedures Subjects completed the task individually. Digits were presented on a computer display in 18 pt Arial bold on a white background. Subjects sat approximately 24 inches away from the screen and held a small microphone.

Subjects were told, "You will see two numbers pop up consecutively. Your task is to figure out the number halfway between them, and say the answer into the microphone. Answer as fast as you can." Additionally, before each block, subjects were instructed, "Please give an approximate answer. Do not calculate."

Figure 1a shows the sequence for each problem. The first number appeared in the middle of the screen for 1s, and then the second number appeared for up to 7s. As soon as the subject spoke, the second number changed font style to 'unbold' for 2s as feedback that a response was recorded. During these two seconds, the experimenter entered the subject's response on a keyboard. A blank screen appeared for 500ms between items.

Subjects completed 8 blocks of 22 items balanced to each contain 6 positive, 6 negative, 7 mixed, 2 anchored, and 1 symmetric pair. Presentation order was randomized individually for each subject within each block.

Prior to the recorded trials, subjects completed 20 practice problems and any confusion was clarified.

Results

Figure 3 shows that perfectly anchored and symmetric trials were solved extremely fast, affirming the privileged status of zero in people's representation of the integers. Of greater interest, mixed trials varied systematically according to their position on the STAS such that problems furthest from symmetry and anchored were solved the slowest. Pure positive and negative problems did not show strong evidence of a tuning curve.

Data Preparation Items tainted by microphone malfunction or premature subject response (e.g. "umm") were trimmed, as were outlier response times (> 3SD off the subject's mean) and inaccurate responses (farther than $1/4^{th}$ the total interval size from the correct answer). This removed 17% of the trials (range of 9% to 27%).

Anchored and Symmetric A MANOVA on the five levels of problem type revealed a significant effect; $F(2,14) = 29.70$, $p<.01$. Two planned comparisons showed that anchored and symmetric items were faster than other problem types; $F(1,15) = 55.98$, $p<.01$. Anchored and symmetric types were not significantly different.

STAS Position To demonstrate the tuning curve for the mixed items statistically, perfectly anchored and symmetric pairs were removed, and the mixed items were grouped into three bins according to their position on the STAS. The first bin had almost symmetric (aSymm) items (STAS of -.99 through -.33). The second bin (Far) contained items far from symmetry and anchoring (STAS of -.33 to .33). The third bin held almost anchored (aAnch) items (STAS of .33 to .99). A MANOVA crossed three within-subjects factors: STAS Position (3 bins), interval Size (sizes 8, 10, 12 – size six was removed because it did not have three STAS bins), and Bias (positive versus negative midpoint, shown as left-right of 0 respectively in Figure 3).

There were no significant effects at the $p<0.05$ level. Using a more lenient threshold of $p<0.10$, the analysis showed a main effect of STAS Position; $F(2,7) = 3.35$. Post-hoc contrasts showed that the Far Position was slower than other two; $F(1,8) = 6.22$, $p<.05$, indicating that people had more difficulty as problems were moved away from anchored and symmetric. Interval Size also exerted an effect on reaction time; $F(2,7) = 4.31$, $p<.10$, such that size 8 was faster than size 12; $p<.05$, Bonferroni corrected. There was an interaction between Size and Position; $F(4,5) = 4.40$, $p<.10$, such that for sizes 10 and 12, the Far was slower than aSymm and aAnch, but for size 8, the aSymm position was slower than Far; $F(1,8) = 17.81$, $p<.01$. There was no effect of Bias.

Pure v. Mixed If the tuning curve involves symmetry about zero, the pure positive and pure negative bisections should not exhibit a tuning curve. A 3 x 2 x 3 MANOVA of STAS Position, Purity (mixed versus positive/negative, or "pure", items), and interval Size showed a main effect of Purity; $F(1,15) = 5.18$, $p<.05$, such that pure items were faster than mixed. There was also a main effect of Size; $F(2,14) = 7.94$, $p<.01$, such that size 8 was faster than size 12; $p<.01$, Bonferroni corrected. The Position factor did not show a significant main effect, but did interact marginally with Purity; $F(2,14) = 2.87$, $p<.10$ -- for mixed trials, response times were relatively slower for Far than aAnch, while for pure trials, Far was no different from aAnch; $F(1,15) = 6.10$, $p<.05$.

Discussion

The results align with the hypothesis that adults have developed a mental representation that semantically encodes the symmetry and identity structures available in the integers. People were faster to respond to anchored and symmetric problems than all other types, showing the privileged status of zero in simple computation. This, however, does not demonstrate a semantic or analog representation. This was provided by the promissory evidence that the ease of problems was determined by proximity to the symmetric and anchored pairs, even when zero was not a midpoint or endpoint. Finally, the pure positive and negative pairs did not exhibit strong evidence of a tuning curve. This shows that the tuning curve disappears when zero is not a possible symmetry point.

At the same time, the data are muddy. One problem in the method was the oral response. For example, subjects may have started to say "negative" as a way to buy time while finding the mid-point. This would distort the true estimate of processing time. A second problem was that some subjects may have calculated rather than approximated the mid-point, despite instructions to the contrary. If so, calculating might contaminate results, as it is theorized to be a verbally mediated process reliant on memorized facts rather than a representation of quantity (Dehaene, Spelke, Pinel, & Stanescu., 1999). Finally, some of the intervals were too small to find evidence of a response curve. Study 2 addresses these concerns.

Study 2: Button-Press Response

In study 2, subjects responded via button press rather than vocalization (Fig. 1b). Also, subjects were randomly assigned to either an approximate or calculate condition.

Methods

Participants Forty right-handed, private university students participated (20 female, mean age = 22.11 years, $SD = 5.65$).

Stimuli The stimulus set consisted of 75 unique pairs, presented twice for a total of 150 pairs. Within-pair interval sizes were 10 (42 items), 12 (50 items), and 14 (58 items). Numbers ranged from -21 to 21. Across the entire stimulus set, each integer appeared between 2 and 12 times, with integers at the middle of the range appearing more often than those at the ends. There were 36 positive and 36 negative pairs, 60 mixed pairs, 6 symmetric pairs, and 12 anchored pairs. For each trial, there was also a target integer that subjects had to judge as close to the midpoint of the stimulus pair or not. Target numbers were either: the midpoint (correct), or the midpoint plus/minus ~1/4 the interval size (to be rejected as implausible).

Procedures There were two conditions**.** In the Calculate condition, subjects received the following instructions on the screen and verbally: "Two numbers will appear. As fast as you can, calculate the number halfway between them. A formula you can use is (a+b)/2." In the Approximate condition, the instructions were: "Two numbers will appear. As fast as you can, think of a number approximately halfway between them. Do this without calculating."

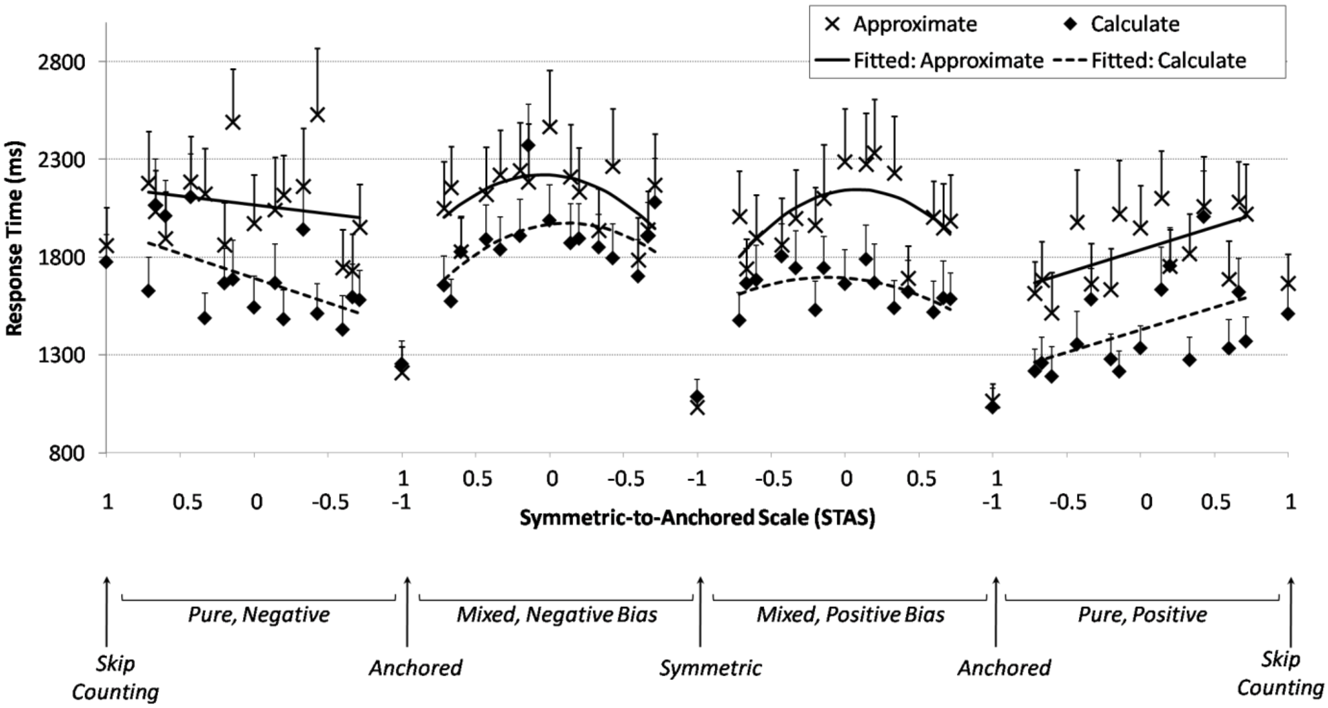

Figure 4: Study 2 response times by STAS value and problem type. X markers and solid fitted lines represent data from the Approximate condition. Diamond markers and broken fitted lines represent data from the Calculate condition. Fitted lines come from multiple regressions on the group means at each response point. Lines were estimated for each group and problem type separately, and are curvilinear if the squared regressor in the model was significant at $p<0.15$. Error bars represent 1 s.e. between-subjects.

The experiment consisted of six blocks of 25 items, with item type balanced between blocks. Subjects sat ~24" from the screen of a Lenovo T400 laptop with 18 pt bold white Arial font on black background.

Subjects controlled the pace of the experiment. For each item (Figure 1b), a fixation cross appeared for 1s, followed by a 400ms blank. Then a stimulus pair appeared and subjects had up to 5s to press the spacebar indicating they had a midpoint in mind. The target appeared immediately after the spacebar depression and subjects had up to 1,250 ms to press Y or N as to whether they thought the target was the actual midpoint. Subjects responded with their right hand only. Y and N buttons were denoted with sticky note labels on the 'l' or ''' (apostrophe) keys on the keyboard.

Before the experiment the subjects completed two practice runs of 12 trials. The experimenter asked the subjects their strategy periodically during the practice runs to ensure they followed the task instructions.

Results and Discussion

Figure 4 shows strong evidence of a tuning curve for mixed trials. In contrast, pure trials show little evidence of the tell-tale arches.

Anchored and Symmetric Outliers were removed as before, with an average of 18% trimmed trials (range 3% to 59%). A 2 x 5 MANOVA with Group (Approximate, Calculate) as a between-subjects factor and Problem Type as a within-subjects factor showed a significant main effect of Problem Type; $F(2,37) = 53.99$, $p<.01$, so that anchored and symmetric problems were significantly faster from the others; $F(1,38) = 93.55$, $p<.01$. Symmetric problems were marginally faster than anchored problems in this study; $F(1,38) = 4.00$, $p<.10$. There was also an interaction by condition such that the Approximate group was slower on the problems that did not involve zero than the Calculate group; $F(2,37) = 3.56$, $p<.05$. Evidently, symmetric and anchored answers were retrieved before the approximation process is completed.

STAS Position A MANOVA used the within-subjects factors of STAS Position (aSymm, Far, aAnch), interval Size (10, 12, 14), and Bias (positive, negative), with Group as a between-subjects factor. There was a main effect of STAS Position; $F(2,26) = 12.55$, $p<.01$, such that the Far position was slower than aSymm or aAnch $F(1,27) = 25.25$, $p<.01$. Thus, the top of each "arch" in the data was significantly higher than the bottoms.

There was also a main effect of Bias; $F(1,27) = 28.96$, $p<.01$, such that negative bias problems were slower than the positive bias problems. The Group and Size main effects were not statistically significant.

Importantly, there was a significant interaction between Group and Position; $F(2,26) = 3.39$, $p<.05$. Planned comparisons showed that the difference between the Far and aSymm/aAnch items was greater for the Approximate group than for the Calculate group; $F(1,27) = 4.24$, $p<.05$. In other words, the response time "arch" pattern was flatter for the Calculate group. The interaction between Group and Position indicates that subjects in the Calculate and Approximate groups were using different mental processes to complete the problems; thus, we successfully controlled the participants' strategies through task instructions. It also shows that when people are encouraged to calculate the answer symbolically, the tuning curve mitigates, as would be anticipated if people were not relying as much on a semantic representation.

Pure v. Mixed A 2 x 3 x 2 MANOVA crossed Purity, STAS Position, and Group. Of particular relevance, there was a significant interaction between Purity and Position; $F(2,37) = 15.13$, $p<.01$. For pure positive and pure negative problems in both groups, response times increased with the absolute magnitude of the problem rather than showing the arching pattern of the mixed problems. This suggests that there is something particular about zero that privileges it in the integer representation.

General Discussion

In an integer bisection task, subjects were very fast to respond to intervals bounded or bisected by zero. Furthermore, response times varied systematically as a function of proximity to these fast zero problems.

The results indicate that zero is a salient feature in the adult mental representation of integers. The behavioral patterns fit an interpretation of zero as a cognitive reference point (Rosch, 1975) that subjects use as a jumping-off station for interpreting or manipulating surrounding integers. The fact that the reference point also facilitates symmetry judgment is not entailed by Rosch's theory. We propose that adults have developed a semantic, analog representation of integers that directly codes symmetry.

An alternative explanation might be that the results are due to differential experience with the problems presented. However, it is hard to make the case that people are faster to bisect -7 and 5 than -8 and 4 because of more experience with -7 and 5. Moreover, if over-learning were the source of the effect, then easy skip-counting problems like 5 and 15 should have also exhibited exceptionally fast responses, which they did not.

Our leading hypothesis is that these adults developed an analog representation of the integers that incorporated important structures of the integer system. It is interesting to consider how this representation of abstract quantities develops. The extremely fast responses for the anchored and symmetric problems may provide one clue. For the symmetric problems, people probably capitalized on a feature of the integer symbol system. They noted that the two digits were identical except for the difference of the negative sign, and they quickly retrieved that the answer has to be 0. Similarly for the anchored problems, people saw that one of the digits was 0, so they simply retrieved the well-learned "half-of" facts. In both cases, the structure of the symbol system privileges zero in finding symmetry or half-way points. Over time, it is possible that these symbolic affordances recruit perceptual processes associated with symmetry. For instance, the symbolic structure makes zero a special point, and people may rely on this point when working with number lines or other extended external representations. Given zero, people can then apply perceptual process associated with finding midpoints, and with practice, these processes may be encoded directly into the internal representation of integers.

In conclusion, we found support for the assertion that the adult representation of integers relies on zero as a structural pillar that aids operations with the set of integers. This may not be a special feature of the quantity zero per se, but rather it may be that zero is an indicator of symmetry in the integer system. Further understanding of how people recruit new computational structures for handling integers will benefit from investigations into the neural underpinnings of the presently observed effect and behavioral investigations of other features of the representation.

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