Blair, K. P., & Schwartz, D. L. (2012). A value of concrete learning materials in adolescence. In Reyna, V. F., Chapman, S., Dougherty, M., & Confrey, J. (Eds.). *The adolescent brain: Learning, reasoning, and decision making* (pp. 95-122). Washington, DC: American Psychological Association.

A Value of Concrete Learning Materials in Adolescence

Kristen P. Blair & Daniel L. Schwartz

School of Education, Stanford University

Corresponding Author:

Kristen Blair

kpilner@stanford.edu

Abstract

There are disagreements about the appropriate role of concrete materials for learning. We clarify the assumptions underlying different uses of concrete materials and abstract symbols in instruction for learning new mathematical ideas. We argue that despite the potential structural isomorphism of concrete materials and abstract representations, they engage different psychological processes. We describe several studies highlighting the unique psychological properties of concrete materials and abstract symbols for learning. We propose the co-evolution hypothesis, where concrete materials and symbols work together to help students discover the structure both in the world and in the abstract representations. In a classroom study with sixth grade students learning about proportion and ratio, students who received instruction consistent with the co-evolution hypothesis showed improved initial learning and greater transfer than students who first learned abstract symbols and mathematical principles, and then applied them to concrete materials as practice.

Most parents of young children would testify to the value of concrete materials for learning. In a well-stocked home, children have hands-on educational toys, books with rich illustrations, and colorful interactive electronics. Even so, there is confusion amongst parents, teachers, policymakers, and instructional designers over the exact value of concrete materials for learning. For example, in California, the state's science curriculum commission proposed legislation that would limit hands-on learning to "no more than 20 to 25 percent" of instructional time. This resulted in an outcry from educators and business people, and the final legislation reversed the proposal to "at least 20 to 25 percent" of science instruction should use hands-on material (emphasis added, 2004, www.cascience.org/csta/leg_criteria.asp).

Confusions about concrete materials take on special importance as children move into adolescence. Adolescents have begun to master the symbol systems of language and mathematics. These children can now learn in other ways besides directly interacting with concrete materials. Moreover, students need to increase fluency in the symbol systems that define adult life. Is there still a place for concrete materials, and if so, how should an educator make decisions?

Ideally, an educator could look to the research literature to find answers. However, the literature is not as clarifying as one might hope. One reason is that there has been a tendency to pit concrete and symbolic materials against one another. Consider the case of mathematics learning, which is the focus of this chapter. Many of the mathematical ideas developed in early adolescence can be expressed both in concrete and symbolic form. For example, fraction addition can be accomplished by manipulating symbols or plastic wedges. This equivalence can naturally lead to an either-or mindset and often generates the question of, "which is better for learning?"

Rather than considering concrete and abstract as a dichotomy, we consider how they can work together to help students learn new mathematical concepts. While concrete and symbolic forms can achieve the same quantitative answers, they have different psychological properties for learning and problem solving. We propose a *co-evolution hypothesis*, where the understanding of concrete and symbolic materials develops in tandem by building on each other's psychological advantages. Of course, not just any combination of materials and activities will lead to optimal learning, and our burden is to help clarify when and how to use concrete materials. (Note that when we talk about concrete materials, we do not only mean physical materials. Our intent is to distinguish between symbolic notations and concrete instances, so that photographs, for example, would also count as concrete materials.)

We begin by briefly reviewing the conflicting research on the value of concrete materials for learning and transfer. We then consider some of the implicit assumptions that guide different uses of concrete materials. Many of these assumptions do not apply to our goal, which is to help students learn fundamentally new (to them) mathematical structures. To clarify this alternative goal and the co-evolution hypothesis, we offer several examples including an explicit instructional example that cashes out the theoretical work in a successful classroom application.

All of the following examples focus on the grouping of quantities, as in the cases of statistical distributions, ratio, fractions, and place value. Learning to work with groups rather than just singletons is a precursor of rational number reasoning for younger students (Blair et. al., 2008; Blair, 2009), and the ability to reason about proportion and ratio is one of the primary accomplishments of pre- to early adolescence and a cornerstone of algebra (Confrey, this volume).

Confusions about Concreteness

Learning from concrete materials has had four primary lines of investigation: motivation, development, transfer, and problem solving. We begin with motivation. Many teachers view concrete materials as engaging (Moyer, 2001). Whether it is the concreteness *per se* or the relief from more typical school tasks is unclear, but there is multi-study evidence that concrete materials improve student attitudes towards mathematics (Sowell, 1989). Surprisingly, to our knowledge, there is no theory of motivation that would predict the special motivation of concrete materials, though nearly all theories could presumably account for the effects after the fact.

In the context of development, the terms concrete and abstract often refer to mental structures and operations rather than external artifacts. Piaget (1941), as well as Bruner (1996), proposed a developmental progression from concrete thought to more abstract understanding. For example, an adolescent can reason abstractly about physical materials, such as making decisions based on non-observable properties or considering hypothetical actions. In contrast, the reasoning of a young child may be more tied to perceivable properties and physical manipulation of the materials.

More recently, theorists have argued that all mathematical ideas and operations, whether symbolic or not, and regardless of age, are grounded in conceptual metaphors that build from sensory-motor activities (e.g., Barsalou, 1999; Glenberg, Gutierrez, Levin, Japuntich, & Kashak, 2004). From this "embodied cognition" perspective, the notion of a set is based on a metaphor of containership, which itself is grounded in the concrete experience of something being inside or outside one's own body (Lakoff & Nunez, 2000).

In contrast to learning progressions that build understanding primarily upon sensory-motor experiences, Vygotsky (1978) emphasized that culturally organized "scientific concepts" drive cognitive development. By scientific concepts, Vygotsky meant a culture's accumulated symbolic knowledge, which is passed from adults to children. Scientific concepts are differentiated from everyday concepts, which stem from concrete experience. From this, one might derive that children should receive cultural forms (e.g., symbolic organizations of thought) to propel learning. Here, we begin to see the theoretical complexity. From many contemporary theoretical perspectives, one might derive that children should receive concrete materials early in instruction. From Vygotsky, one recognizes the importance of culturally organized symbols in driving initial learning. We are left without a clear prescription of what the interaction should be between concrete and symbolic materials for development.

Transfer research has also yielded conflicting results and does clarify the most appropriate use of concrete materials. Transfer research asks whether students can use their learning in new contexts that differ from the conditions of original learning. For example, if children learn about fractions using pie pieces, can they transfer to solve problems with tiles? Some studies have found decreased transfer when learning occurs with concrete instances compared to only abstract symbols (e.g., Bassok & Holyoak, 1989; Kaminksi & Slotsky, this volume; Kaminski, Sloutsky, & Heckler, 2008), while others have shown improved transfer for concrete materials over abstract ones (e.g., Goldstone & Son, 2005; Kellman, Massey, & Son, 2010; Schwartz , Chase, Chin, & Oppezzo, in review). In the final discussion, we attempt to reconcile some of the conflicting results from the transfer literature.

Another source of confusion about the value of concrete materials comes from the literature on problem solving. Effective problem solving is one of the main precursors to learning (Anderson, 1982). Ensuring that students succeed at problem solving is a major way educators can help students learn. Some scholars propose that concrete materials help students connect new ideas to their prior knowledge. In turn, students can make sense of complex mathematical problems that they simply could not handle as naked equations (Barron et al., 1998; Driscoll, 1983; Resnick, 1983; Sowell, 1989; Koedinger & Nathan, 2004). In contrast, some researchers have found that students focus on irrelevant aspects of the concrete materials (Harp & Mayer, 1998; Kaminski et al, 2009), and this pulls them away from the abstract mathematical structure. More generally, concrete materials are subject to multiple interpretations, and children may not find the designer's intent in the materials. For example, where the teacher sees groups of ten being manipulated, the student may see only many individual items (Thopmson, 1994; Blair, 2009). At an extreme, concrete materials may drive concrete ways of thinking instead of scaffolding more general symbolic interpretations (Schwartz & Moore, 1998; Uttal, Liu, & DeLoache, 1999).

Clarifying Assumptions on the Relation of Concrete and Symbolic Materials

The conflicting research and theories may be confusing for teachers and curriculum designers who try to decide whether and when to use concrete materials. It may be useful to consider the assumptions that drive different proposed relations between concrete and abstract materials. Often these assumptions are tacit, and educators and researchers may not be aware they are making them. A key assumption involves what students already understand prior to using the materials. Table 1 lays out different possibilities of student understanding and their implications for the relation of concrete and symbolic materials in learning.

[Table 1 about here]

Practice

The upper-left cell of Table 1 holds practice activities. The assumption is that students already understand the concrete materials and the symbols used to represent them. Using them together strengthens their association and the automaticity of mathematical thinking.

Concrete → Symbolic

The lower left cell of the figure reflects a very common assumption in the use of concrete materials. Namely, students already understand the concrete materials, and through instruction, they can use the concrete materials as meaningful anchors for more obscure symbolic representations. For example, students might be learning the term “1/4.” To make this meaningful, one might have the students work with a circle made of four equal wedges. The students could pull out one wedge, and it would be labeled “1/4.” The student could then pull out a second wedge, and the two wedges together would be labeled “2/4.” From this perspective, students are not learning fundamentally new concepts through the use of concrete materials. The assumption is that they already understand what it means to have a physical whole split into parts, and what it means to count up the parts of the whole. Instead, students are learning to ground symbolic operations and notations with physical materials that they already comprehend.

Using relatively familiar concrete materials and actions to ground symbolic terms and operations makes a good deal of sense. Getting to see a zebra surely helps understand the word “zebra.” However, in the case of learning new concepts, the assumption that students can map from a known concrete instance to a novel symbolic representation does not always apply. This is because beginners often do not see the same structure in concrete instances as experts (e.g., Gibson, 1969; Goodwin, 1994; Marton & Booth, 1997). Expert radiologists, for example, can see diagnostic details in x-rays overlooked by residents (Myles-Worsley, Johnston, & Simmons, 1988).

When learning new mathematical concepts, there is a risk that students will not see the intended structure in the concrete materials, and therefore, the materials will not effectively ground the symbolic representations (e.g., Thompson, 1994). For example, in one class, we showed students a manipulative they had worked with for several days. The manipulative is intended to instantiate place-value notation in concrete form. We held up a single small cube, and asked the students how many. They said, "One!" We then held up a stick of ten cubes, and they said, "Ten!" For a flat made of ten sticks, they said, "A hundred." Finally, we held up a large cube comprised of ten flats. Many said, "six hundred." Despite having handled the heavy cube before, they perceived that it had six sides of one-hundred flats. Asking students to map new symbolic structures to poorly encoded concrete materials will not yield much success for understanding.

Symbolic → Concrete

Scientists regularly use known symbolic procedures and structures to make sense of concrete instances. For example, given the problem of understanding the behavior of water in a tilting glass, scientists might try different integrals to see which provides the best model. Giving students opportunities to use symbolic math to explain novel concrete instances is a good way to develop scientific sensibilities. However, it is important to distinguish the conditions of mature performance from those of early learning. In the example above, the scientists already understand the structure of the symbolic formulas used to characterize the phenomenon. Students learning a new concept may not yet have a deep understanding of the symbolic procedures they are taught.

The symbolic → concrete cell would suggest that students should memorize or derive symbolic procedures and structures first, and then apply them to novel concrete instances. Arguments in favor of this approach are that learning abstract symbolic procedures will lead to better transfer, because learning will not be too heavily tied to particular concrete instances (e.g., Kaminksi & Slotsky, this volume). However, the risk of this approach is that students will rely on the symbolic procedures, and never learn to see the structure in the concrete materials, leaving them with unstable symbolic rules (e.g., Hiebert & Wearne, 1996). In a later section, we provide an example of this problem. Students focus on the application of the formulas at the expense of learning the underlying contextual structures that make the formulas applicable in the first place. This hinders both initial learning and transfer.

Co-Evolution

Thus far, the relations between symbols and concrete materials are predicated on an isomorphic, one-to-one mapping between them (Post, 1981). As observed earlier, people can often manipulate physical objects and symbols to achieve the same answer. This leads to the assumption that learning involves mapping the correspondences between concrete and symbolic operations. The potential formal isomorphism between concrete and symbolic materials, however, does not mean there is a psychological isomorphism. Concrete and symbolic understandings have different properties, and learning new ideas depends on leveraging these properties in relation to each other to achieve a deeper understanding than either could achieve alone.

The co-evolution cell proposes that concrete materials and symbols work together to help students discover the structure both in the world and in the abstract representations. We say more about this hypothesis below. But pending that elaboration, a concrete example of co-evolution might help. We develop the example using a model of instruction called, “inventing to prepare for learning” (IPL).

IPL asks students to invent symbolic accounts to characterize concrete instances. Our current example involves students learning about variance (for a representative collection of papers on learning variance, see Lajoie, 1998). The students who received the instruction had some vague intuitions about variability, but there was little precision. At the same time, the students understood basic arithmetic, but they did not know how to compute variability. The IPL instruction encouraged students to co-evolve both types of understanding.

In a series of studies, 9th-grade students learned statistical concepts and uses of variability (Schwartz & Martin, 2004). As a fairly prototypical activity, students used math to invent a way to compute an "index" that could be applied to different pitching machines to let customers know the reliability of each machine. The narrative is that different companies produce baseball pitching machines. Figure 1 shows the results of testing the pitching machines. The black circles show where a ball landed when aimed at the X in the center. Students had to invent a "consumer index" that indicates the relative "consistency" of the different pitching machines.

[Figure 1 about here]

The grids were designed as contrasting cases (Bransford & Schwartz, 1999). Contrasting cases, much like wines side-by-side, can help novices perceive structure they might not already know. For example, in "Smyth's Finest," the balls are tightly clustered but far away from the target. This was intended to help students differentiate "inaccuracy" from "variability," which students often conflate. The students also had to invent an indexing procedure that would work across the cases. Here, the idea is that a demand for a precise symbolic, quantitative account leads them to look more carefully at the concrete instances, and find the common structure across them. So, for example, students have to pay attention to the different numbers of balls used for the different pitching machines.

While doing the inventing activities, students made progress, but most did not come to a standard solution for capturing variability. However, after the inventing activities, students were more prepared to learn standard symbolic formulations and the situations they have been designed to describe. For example, the students received a short lecture on the mean deviation formula and practiced for a few minutes. A year later, these students were more capable of explaining why variance formulas divide by 'n' (e.g., "To find the average"), than were college students who had recently taken a semester of statistics (e.g., "Doesn't it have something to do with degrees of freedom?").

With respect to the co-evolution hypothesis, students began with a vague, undifferentiated understanding of variability. In terms of concrete instances, they did not initially recognize a difference between inaccuracy and variability, and they did not consider the number of samples at all. In terms of symbolic understanding, the students did not initially have any knowledge of notations like Σ, or the idea of variability as the average of the differences between the samples and the mean. Yet, by using symbols and concrete materials together, they developed an understanding of the concrete materials and symbolic operations. This prepared them to learn from subsequent direct instruction that introduced formal concepts of variability and their symbolic notations. In the co-evolution process, the relationship between the concrete materials and symbolic formulas was not psychologically isomorphic, it was synergistic.

To unpack these ideas in the following sections, we first detail the psychological value of concrete activity for learning new ideas and the importance of learning to perceive mathematical structure in concrete materials. Then, we move to the psychological value of symbols for helping drive precise perception and generalization. Finally, we present a study, much like the preceding statistics study, that more directly highlights positive and negative possibilities for interactions between symbolic and concrete materials when learning new ideas.

The Value of Concrete Activity for Learning New Mathematical Structures

Overcoming prior knowledge

We begin with two propositions. (1) A major challenge of learning to perceive new mathematical structure is that people will see what they already know rather than possibilities for what is new. (2) Interacting with concrete materials can help. The following example demonstrates these two propositions.

[Figure 2 about here]

Martin and Schwartz (2005) worked with 4^{th}-grade students who were at the cusp of understanding fraction operations. In the picture condition, the children saw a collection of pieces much like those shown in the left panel of Figure 2. Children had to circle a subset of the shapes to indicate their answers to problems such as, “what is 1/4 of these 8 pieces?” In the manipulate condition, the same children received the pieces rather than a picture, and they had to move the pieces to show the answer. Each child completed both conditions twice using a variety of simple problems (e.g., 1/3 of 6 pieces). Regardless of order, the children were roughly three times more accurate in the manipulate condition than the picture condition.

What explains these results? In the picture condition, the children relied on their well-practiced, natural number schema. For example, they would circle 1 chip, 4 chips, or both 1 and 4 chips. They interpreted the ‘1/4’ as referring to the natural numbers ‘1’ and a ‘4’. This reliance on prior knowledge was quite strong in the picture condition. In fact, in a second study, the pieces were already pre-grouped as shown in the right panel of Figure 2. The students made the same mistake, circling 1 or 4 chips. The students simply did not see the intended meaning of this grouping organization.

In contrast, when the children had an opportunity to manipulate the pieces, they saw new possibilities emerge. For example, children moved several pieces at the same time, and this may have helped them recognize that pieces that moved together could be counted as one group. Once they started to perceive the relational possibility of grouping, they were on their way to solving the problem. For example, in a typical instance, children would start making separate piles of chips, and then they would stumble on the idea of making them equal groups. Once they had equal groups, they realized they could pick one of them as the answer. Manipulating the environment helped release the children from an over-reliance on old interpretations to help them develop new ones (Martin, 2009).

Overcoming a lack of prior knowledge

One of the compelling aspects of concrete materials is that they show the concrete outcome of an action. For example, if people do not throw a ball hard enough, it falls short of the target. In contrast, if people make a mistake with a physics equation, the erroneous outcome will only become apparent if people check their equations, get told they are wrong, or apply the equations to an actual instance. The ready availability of outcomes and feedback creates an intuitive argument for the value of concrete interactions during learning.

At the same time, students may not have sufficient prior knowledge interpret the feedback well. The feedback literature tacitly presupposes that informative feedback is readily perceived, and any problems occur in the internal processing of that information (Blair, 2009). However, based on our analysis of the difficulty of perceiving novel structure in concrete materials, the same problem should apply to feedback. People may rely on coarse aspects of feedback they already understand and miss more informative details. For example, a novice golfer may notice that his ball overshot the hole, but fail to notice that the ball did not have enough backspin.

Given the potentially major omission in the feedback literature, Blair (2009) examined how children learn to pick-up structural information in concrete feedback. Fourth-grade students used a computer game called *Spiderkid* to learn about iterated units (groups) in the context of bases and place value. We take some space to explain the game, before describing the results.

The game was designed as a Teachable Agent (e.g., Chin et al., 2010), where students learn by teaching a computer character. In this case, students taught Spiderkid how to use spider webs to make rescues from the city's skyscrapers. (His Uncle Spiderman is planning on retiring.) The floors of the city's buildings are designed with a recursive structure so that on different tries, the buildings could map into the bases of 3, 4, 5, or 6. For example, a base-3 building has special marks for each floor (3^0), for every third floor (3^1), and for every ninth floor (3^2).

Spiderkid has 3 kinds of webs, which can each be set to go a different distance. For example, to match the base-3 building, the student should set Spiderkid's webs to shoot 1, 3, and 9 floors, respectively. The goal is for children to choose the right size and numbers of webs for Spiderkid to rescue a cat trapped on one of the floors. The game proceeds through levels where students solve increasingly difficult problems. At each level children get to see what Spiderkid does based on what they have taught him.

As an example of how the feedback worked, Figure 3 shows a problem using a base-5 building. It is an early problem for the child, because it has been simplified to only have floors marked at 5^0 and 5^1, and the child is "training" Spiderkid before he is ready to rescue cats. The child has entered the size of the web that she believes will get Spiderkid to only land on the floors indicated by the thicker lines in the left building. The child has incorrectly entered "2." The right building shows the outcome, where Spiderkid does as he was told. As feedback to the student, Spiderkid animates climbing the building using webs of size 2. As he climbs, Spiderkid leaves a trace for each floor he lands on. The child can then modify her answer in response to the feedback until Spiderkid matches the building on the left exactly. For this problem, the child repeats the same process of teaching Spiderkid for the place values of 5^0, 5^1, and 5^2. To get to the next level of the game, students have to get two consecutive problems correct on the first try.

[Figure 3 about here]

At more advanced levels of the game, the children have to choose the sizes of all three webs. Finally, once Spiderkid has completed the training, the children are ready to rescue the cat. As shown in Figure 4, they have to indicate the length of each web, and how many Spiderkid will need of each. For example, to rescue a cat on the 17^{th} floor of a base-3 building, children should optimally choose one 9-floor web, two 3-floor webs, and two one-floor webs (children learn that it is faster to use long webs first, so they do not just choose 17 1-floor webs). For the children, this corresponds to specifying the "place" and "face" values for a given quantity, and children do this for base-3, -4, -5, and -6 buildings.

[Figure 4 about here]

In one study, 9-10 year-old children played Spiderkid for an hour, producing a log file of their actions and outcomes. A specially designed software tool (Blair, 2009) analyzed the log files by tracking student responses to feedback over time. Based on the students' post-feedback adjustments, the tool inferred what information the students extracted from the feedback. There were four levels of information. (1) *Correct/incorrect information*. Students only saw that Spiderkid was wrong. On their next turn, they changed their answer, but they actually made things worse by adjusting the wrong direction. (2) *Direction information.* Students saw whether Spiderkid had jumps that were too big or too small. On their next turn, children corrected the webs in the right direction, so they got closer by a bit. (3) *Approximate magnitude information*. Students saw both the direction of their mistake and whether it was a big or small mistake. On their next turn, children made a large correction in the right direction. (4) *Exact magnitude.* Students saw that they were off by a precise amount. On their next turn, children corrected their teaching by the exact amount of the discrepancy from the prior turn. (There was also a "No Valid Information" coding, which occurred when the student did not appear to gain any information from the feedback, such as failing to change an incorrect response at all.)

[Table 2 about here]

Table 2 shows the percentage of times children transitioned between levels of feedback perception from one try to the next. For example, the 83% in the lower-right corner indicates that once children learned to count the exact magnitude, they continued to do so for the next attempt 83% of the time. The table indicates that children generally increased the level of information they extracted from the feedback (i.e., larger percentages are above the diagonal).

Not all students followed this progression, however, and those who were unable to learn to see the information in their mistakes made little progress. For example, after playing the game, children were asked to draw what the buildings had looked like. Students who were characterized as being at lower levels of feedback perception failed to draw the equal interval grouping structure of the building in the redrawing task, and instead drew floors haphazardly. These students did not appear to perceive the equal interval structure of Spiderkid's webs as he gave the student feedback, so they only noticed that they were wrong and nothing more.

A second study analyzed the pointing and speech of a new group of children as they interacted with the environment. The protocol data indicated that the children were learning to notice more precision in the feedback. For example, across several early trials, one student's comments only referenced approximate magnitude information as she watched the feedback; "oh, that's too much" or "too little". It was not until later that the student started to notice the exact amounts in the feedback, and then eventually found the base-structure with the exclamation, "I can multiply – it's 36."

In summary, interacting with concrete materials can help students come to appreciate new (to them) quantitative structures. However, concrete materials are not simply "read off" by novices. Students must come to interpret the mathematical relations they involve, and interacting with the materials can help. In the case of learning fractions, children did better when they could manipulate the concrete materials. In the case of learning place values, children did not initially see the quantitative structure in the buildings or the feedback. Over time, they began to see the quantitative information more precisely. At the same time, some students did not find the mathematical structure in the concrete materials, even with Spiderkid, where the children received quantitative feedback that showed how their actions fell short of the outcome. Concrete materials can only achieve so much, which is where symbols come in.

The Value of Symbols for Learning New Structure

Generating symbolic accounts of concrete materials

When learning with well-designed concrete materials, students can apply psychological processes that help them discern more structure in those materials. What learning processes do symbolic materials support? Abstract, symbolic notations are often taken as a way to compute an answer. However, they have other psychological properties that can help children learn to perceive structure in instances, particularly relational structure. Consider the case of the balance scales shown in Figure 5. In this classic developmental task (it has also been used with adults, e.g., Shen, 2006), children decide if the scale will balance, tilt left, or tilt right. The left side shows a scale with quantities that are very hard to count, whereas the right scale makes it much easier to count and to turn the physical quantities into discrete symbolic terms (e.g., "three" weights, "one" peg).

[Figure 5 about here.]

Ten- and 11-year-old students worked with several problems of either type (Schwartz, Martin, & Pfaffman, 2005). The hypothesis was that students working on the hard-to-count quantities would do quite poorly, because they could not enlist the aid of symbolic numbers. The hypothesis was supported. Almost 70% of the fifth-grade students in the hard-to-quantify condition performed at the level of sophistication generally associated with 5-year-olds when using a countable balance scale (Siegler, 1981). They reasoned exclusively about the weight of the beakers. They even failed to reason about the distances when weight of the beakers was held constant. For example, when two beakers were identically filled, they would say the scale would balance, even if the beaker on one side was at the very end and the other beaker was right next to the fulcrum. In contrast, only 30% of students in the countable materials condition performed like 5-year-olds. Instead, they performed at their age level.

Why did the easy-to-count condition help? These children did not have a firm grasp of the multiplicative relationship (weight x distance), as indicated by the performances of the otherwise similar children in the hard-to-count condition. There were at least three ways that using abstract mathematical symbols helped. (1) Children could combine perceptually distinct dimensions, for example, weight and distance. A '3' can refer to a distance or a weight, and by representing weight and distance with digits, it made it possible to put them into a quantitative relationship that would be hard to do perceptually. (2) The symbols provided a compact representation that alleviated working memory burdens, at least compared to trying to maintain mental images of weight, distance, and the relation between them. (3) Arithmetic provides a set of possibilities for generating possible explanations. Much like children's hands provided easy ways for children to manipulate the tile pieces in the fraction study above, simple arithmetic provides a set of candidate moves. For example, children can try adding values, multiplying values, and so forth. Different symbolic actions can help spark new interpretations.

A second set of studies more clearly demonstrates that the symbolic structure of math can propel the learning of proportional relations (Schwartz, Martin, & Pfaffman, 2005). Eight- and

10-year-olds worked with the easy-to-count balance problems in an online environment. Students received a series of problems that ranged in difficulty. For each problem, students predicted whether the scale would balance, tip right, or tip left, when it was released. Students were asked to justify their predictions in a little text box on the screen. In the invent-math condition, children had to use symbolic math to justify their answers (e.g., "3 > 2"). In the words-only condition, children had to use words to justify their answer (e.g., "the left has more"). This was the sole difference between conditions. After answering, the children saw an animation of what actually happened when the scale was let go, and then moved on to the next problem.

On a posttest with novel problems, children in the words-only condition performed at their age norms. They tended to focus their justifications on either weight or distance, but not both. In contrast, students in the invent-math condition performed above their age norms. The 8-year-olds came up with answers that tried to integrate both weight and distance, and the 10-year-olds ended up solving the problems as well as adults by the posttest.

A fairly typical prototypical sequence in the invent-math condition comes from a girl who eventually found the multiplicative relation. What follows are her mathematical justifications typed into the textbox. (She did not label the values.)

a) 3 > 2

b) 4=4

c) 3+1 > 2-2

d) 3+3 = 4 + 2

e) ???

f) 2 x 3 = 3 x 2.

For problems (a) and (b) she only considered one dimension (either weight or distance). At (c), she started to consider both the weight and distance dimensions in her explanation. At (d) she starts to use a single operation on both sides of the equation. Finally, by (f) she tried multiplication to relate the two dimensions and it worked. While this could appear to be blind symbol pushing, the symbols gave her a way to relate the dimensions of weight and distance and to test out mathematical relationships. This allowed her to discover the multiplicative relationship in the concrete materials.

The invent-math condition did not simply yield lucky trial and error success. It improved the children's qualitative understanding of the balance scale. On a transfer task that involved weights on three pegs, children in the invent-math condition could not solve the problems correctly, but they continued to reason with both the weight and distance dimensions simultaneously. And, as before, the words-only students focused on either weight or distance, but not both. Consistent with fuzzy-trace theory (Reyna & Brainerd, 2008) precise quantitative understanding predated qualitative intuitions.

Creating Effective Instruction for Adolescents

The co-evolution hypothesis proposes that students ideally learn symbolic and concrete structures together. This differs from the one-to-one mapping approach, where students start with one or another, and then map back and forth between concrete and symbolic presentations of quantities. The preceding studies were designed to investigate basic psychological processes in learning, and they were not intended to teach students the best way possible. How do we cash out the insights from these studies to design classroom learning?

The prescription is straightforward at a general level. (1) *Focus on Big ideas.* Promote co-evolution when students are learning new mathematical structures. Not all learning involves learning new concepts; but, for big ideas, it is important to have students develop a strong foundation early. (2) *Optimal concrete instances.* Provide concrete materials designed to help students come to perceive structure and interpret feedback. Our specific approach is to use contrasting cases. We carefully juxtapose concrete instances, where the differences and similarities can help students come to find structure. Plus, by having multiple cases, students can self-generate feedback by seeing if a solution for one instance applies to another. (3) *Inventing Mathematical Structure.* Have students try to formulate symbolic organizations that can account for the structure (or processes) in the concrete materials. We have called this 'inventing' to highlight the fact that students do not begin with a pre-formulated answer. Also, it is important to note that students do not need to actually discover the correct solution through their inventing activity (Kapur, 2010). They simply need to start recognizing the key structures, which prepares them to understand subsequent formal explanations more deeply (Schwartz & Bransford, 1998). So, even if students invent incorrect solutions, they are more prepared to learn later than if they are told the correct solution at the outset (Schwartz & Martin, 2004). This latter point is particularly delicate.

One might think that it is best to provide students with the symbolic formulation upfront to help make sense of concrete materials, rather than have them invent their own. (This would be the symbolic -> concrete cell in Table 1). However, this approach runs into the problem that abstract representations can overshadow student learning of mathematical structure. In particular, students may come to rely on the symbolic procedures, and this will interfere with their ability to find the structure in the concrete materials. The following study demonstrates this point.

To instantiate these high-level prescriptions for adolescent learning, Schwartz et al. (under review) worked with 8^{th}-graders learning physics. At this age, children learn about speed, density, and force. The three tenets were operationalized as follows:

(1) Focus on Big Ideas. Density, speed, and force are separate big idea in physics. Mathematically, however, they all depend on a single big idea – ratio and proportion. The formulas, D=m/V, S=d/T, and F=ma, all comprise intensive ratios among unlike physical quantities. Children of this age can procedurally solve problems involving ratios – they simply need to divide. But, this does not mean they understand ratio structures. For example, they may not spontaneously notice the relevance of ratio in a new problem.

[Figure 6 about here]

(2) Optimal concrete instances. Figure 6a shows a "contrasting cases" worksheet that was specifically designed to help students perceive proportionate ratios while learning about the concept of density. The narrative of the worksheet is that each row represents a company that ships clowns to events. A company always packs its clowns into busses by the same amount, though it may use busses of different sizes. The meaning of "same amount" is what children need to come to understand (i.e., the same density). The degree to which the companies pack their clowns differs across the three companies.

The contrasting cases are designed to include three levels of features. The first level is surface features. In the figure, two examples of surface features are the type of clown and the lines defining the exterior of the busses. These incidental details are irrelevant to the concept of density, and they were included to examine the effect of surface features on learning as well as make the tasks entertaining for the students. A second level of feature is the specific density used by a company for each of its busses. For example, the company shown in the first row has one bus that has 4 clowns and 4 bus compartments, and a second bus has 2 clowns and 2 compartments. The within-company density is a ratio of 1:1. The third and deepest level of structure is the general feature of ratio, which occurs *across* the paired cases or companies. While the specific values of the ratios differ for each company, all three use proportionate ratios. This last level of feature is termed the "invariant under transformation," or the deep structure. The invariant of ratio persists, despite changes in specific densities and surface features.

(3) Inventing Mathematical Structure. Students were told to make a "crowdedness index" that would enable comparison across the cases. There are three reasons students had to make an index. The first is that it is important for students to engage the structure across the cases, rather than take each case one at a time. If students do not try to make an index that covers all the cases, then they will be less likely to perceive the invariant under transformation. Therefore, they are asked to invent a single index procedure that works for all the cases. The second reason is that working towards a compact and quantitatively precise index creates a simultaneous demand for precision when noticing the structure. For example, as demonstrated in the earlier studies with the Spiderkid game, it is not sufficient to just say and notice that one of the cases has "more" or "less" than another. Quantification requires identifying how much more or less. The third reason is that telling students the symbolic solution too soon shortcuts their search for the deep structure in the concrete materials. Inventing is one way to engage mathematical thinking without undermining the search for quantitative structure in the concrete instances.

The studies had two parts. In the first part, students in the *Invent* condition tried to invent a crowdedness index for the clowns in Figure 6a. In the *Tell-and-Practice* condition, students were first taught the concept of density, its formula, and they received an abstract worked example of using the formula that highlighted the proportional structures without potentially distracting details. Their task was to use the formula to find the density of the clowns used by each company. The Tell-and-Practice instruction was meant to be a fair representation of the ubiquitous model of telling students the procedures first and then having them practice on a set of problems.

A day later, the students were asked to redraw the clown worksheet from memory. By looking at what the students remembered, it is possible to determine what structure they found in the contrasting cases. Students' drawings were coded for the number of surface features they recreated (e.g., dotted lines around busses). They were also coded for the pairs of busses that were recreated with the deep structure of a proportionate ratio. Figure 6b provides portions of student drawings that were high and low on surface and deep features. Figure 6c shows the results. Students in the Tell-and-Practice condition did not re-create the deep structure of the clown worksheet as frequently as the Invent students. This failure at encoding was not due to paying less attention to the task, because both conditions showed the same performance at recalling the surface features. Being told the symbolic method before the examples undercut the search for structure, because students could solve the problems symbolically without finding the structure in the concrete instances.

The second part of the research examined whether these differences in instruction and structural encoding had implications for learning physics. As before, the primary manipulation was whether students were told the symbolic formulas and then practiced applying them to concrete instances, or whether they tried to invent symbolic procedures to capture the structure of the materials first, before being told the standard formulas.

After finishing the recall task, the Tell-and-Practice students received a lecture about ratio and the prevalence of ratio concepts in physics, including density, speed, and force. Then, over the next few days they completed tell-and-practice activities for three more sets of contrasting cases that covered density and speed. As before, they received brief explanations of the relevant concept, the formula, and an abstract worked example, and then they practiced applying the formula to sets of contrasting cases.

The Invent condition worked with the same density and speed contrasting cases as the Tell-and-Practice students, but as before, they had to invent an index. They were only told about the standard formulas and concepts after completing all the cases. This was done by giving them the same lecture that the Tell-and-Practice students had received earlier. On the last day of instruction, both groups of students practiced on a set of standard word problems (e.g., find the density given two values).

Several weeks later, the students received two types of posttests. One posttest involved measuring whether students spontaneously transferred ratio to understand the spring constant. (The stiffness of a spring is the ratio of displacement by load.) They received a sheet similar to Figure 7. Students had to develop a way to describe the stiffness of the trampoline fabric, which is an instance of finding a spring constant. (The problem is a simplification of how trampolines work.) On the transfer problem, students in the invent condition were roughly four times as likely to use a ratio to describe the stiffness of the fabric for each of the trampolines. The

Tell-and-Practice students were more likely to describe the stiffness of the fabric with a single number representing only one dimension, for example, the number of people or the number of rungs the fabric stretched, but not both. The strong advantage for the Invent group was replicated, even when the posttest only included a single trampoline instead of four. Notably, the Invent advantage was equally strong for both the high- and low-achieving students.

[Figure 7 about here]

The second type of posttest included more traditional word problems. Students answered problems about density and speed. On this test, the students in both conditions performed equally well. Thus, the inventing activity did not diminish student learning of basic symbolic procedures, and it worked well for students at all levels of achievement.

In summary, the inventing activity helped students perceive the ratio structure of density and speed, which in turn, helped them transfer the concept of ratio to understand new situations. In contrast, the Tell-and-Practice condition led to symbolic proficiency as measured by the word problems. Given the symbolic formulation, these students focused on what they had been told rather than the structure of the concrete situation. They did not see the ratio structure, neither during instruction nor at transfer. In short, being told too soon prevents the co-evolution of concrete and symbolic understanding, and students do not learn the big ideas very well.

Final Thoughts

What role should symbols and concrete materials play in the learning of new concepts? The co-evolution hypothesis suggests that symbolic actions can lead to new interpretations of concrete materials, which in turn can drive deeper understanding of the abstract symbols. However, not all relationships between symbols and concrete materials are optimal. We have argued that abstract symbols should be used in the service of trying to find and capture structure in concrete materials during the initial learning of new big ideas.

Our lead argument is that the psychological processes and benefits of handling symbolic and concrete materials are different, and they should be put in a complementary relation, and we provided specific examples to the point. This perspective differs from a view of mathematics learning that treats the abstraction of mathematical ideas as a process of subtraction. Confrey (1995) labeled this perspective, *trahere*, the Latin term for "throw away." According to trahere, learning involves subtracting away the non-essential surface features and properties of concrete materials. For example, Kaminski and Sloutsky, (this volume) found transfer benefits for presenting mathematical concepts abstractly as compared to embedded in contextual examples. Researchers argue that if general principles become too deeply embedded in a particular context, students will not recognize the abstract mathematical structure and therefore, they will not be able to apply their ideas to new concrete instances that differ on the surface (Bassock & Holyoak; Kaminski et. al., 2008). The assumption appears to be that "real" mathematical understanding is independent of context, and therefore, to accelerate learning, instruction should start with the abstraction to avoid problems with concrete materials.

From our perspective, a key aspect of mathematical learning involves coming to find the quantitative invariants within a concrete instance (Gibson, 1969). Invariants are those properties that generalize to new instances without losing their underlying structure. So, by this perspective, generalization does not depend on subtracting away concrete surface details to get to the essential abstraction. Rather, generalization depends on learning to perceive and account for structure in concrete instances (e.g., see Fruedenthal, 1973; Greeno, Smith, & Moore, 1993).

To achieve this level of insight, students need to coordinate the psychological processes associated with symbolic and concrete materials.

The crucial question for novel learning is not the amount of superficial or contextual information contained in the concrete materials, but rather how students interact with the materials. This may help explain some of the seemingly conflicting results between our studies and those of Kaminski and Sloutsky (this volume). For example, Kaminski and Sloutsky found that people who learned from a concrete instantiation were more likely to reference superficial features and less likely to reference structure when recalling, compared to people who had learned from a more abstract instantiation. In contrast, in the clown density study we found that students who had to invent a symbolic solution for concrete materials recalled the same level of surface detail and more structure than students who had been told the abstract formula first. We believe one reason for our results is the processing orientation the inventing students took. They had to integrate across instances to find the invariant structure within the concrete materials. By this interpretation, participants in the concrete condition of Kaminski and Sloutsky's study may have been able to use prior knowledge to solve each instance independently, shortcutting the need to seek general structure.

To test this hypothesis, Schwartz, Chase, and Bransford (in press) conducted a new study modeled after the clown density study. As before, students either had a Tell-and-Practice or Inventing processing orientation. The new, crossed factor was that half of the students in each condition worked with either abstract or concrete materials (abstract dots or clowns). Consistent with Kaminski and Sloutsky (this volume) in the Tell-and-Practice condition, students who worked with abstract materials performed better than students who worked with the more concrete instantiation. However, in the Inventing condition, where students had to search for structure across the cases, the effect of concrete versus abstract materials disappeared. Moreover, the regardless of the concreteness of the materials, the Inventing students did twice as well as the abstract Tell-and-Practice condition.

Concrete materials support the discovery of new structure, and symbolic materials provide ways for students to account for the invariants in that structure. As the preceding studies have shown, these processes need to work in tandem. It does not work to assume that students already understand symbols and this will be sufficient for applying them to concrete materials. For example, in the final study, we demonstrated that teaching students the symbolic formulation actually shortcuts the search for structure in concrete materials. It also does not work to assume that students can "read off" novel structure from concrete instances. For example, without the support of symbolic activity, students will rely on vague prior knowledge. Thus, the challenge for developing mathematical understanding, in adolescence and all ages, is an instructional problem of how to support co-evolution. The inventing activity over contrasting cases provides one promising solution.

References

Anderson, J. R. (1982). Acquisition of cognitive skill. *Psychological Review, 89*, 369-406.

Barron, B. J., Schwartz, D. L., Vye, N. J., Moore, A., Petrosino, A., Zech, L., Bransford, J. D., & CTGV. (1998). Doing with understanding: Lessons from research on problem- and project-based learning. *Journal of the Learning Sciences, 7*, 271-312.

Barsalou, L. W. (1999). Perceptual symbol systems. *Behavioral and Brain Sciences, 22,* 577–609.

Bassok, M., & Holyoak, K. J. (1989). Interdomain transfer between isomorphic topics in algebra and physics. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 15,* 153-166.

Behr, M., Post, T., Silver, E., & Mierkiewicz, D. (1980, August). Theoretical Foundations for Instructional Research on Rational Numbers. In R. Karplus (Ed.) *Proceedings of Fourth Annual Conference of International Group for Psychology of Mathematics Education* (pp. 60-67). Berkeley, CA: Lawrence Hall of Science.

Blair, K. P. (2009). *The neglected importance of feedback perception in learning: An analysis of children and adults' uptake of quantitative feedback in a mathematics simulation environment*. Ph.D. dissertation, Stanford University, Stanford, CA.

Blair, K., Hartman, K., Martin, L., Hannula, M., Yun, C., McCandliss, B., & Schwartz, D. L. (2008, March). Groupthink. Symposium at the *Annual Meeting of the American Educational Research Association,* NY, NY.

Bransford, J.D., & Schwartz, D. L. (1999). Rethinking transfer: A simple proposal with multiple implications. In A. Iran-Nejad and P. D. Pearson (Eds.) *Review of Research in Education,* *24*, 61-100. Washington, DC: American Educational Research Association.

Bruner, J. (1996). *The Culture of Education*. Cambridge, MA: Harvard University Press.

California Science Teachers Association (2004). K-8 Instructional Materials Adoption. Retrieved from http://www.cascience.org/csta/leg_criteria.asp.

Chin, D. B., Dohamen, I., Oppezzo, M., Cheng, B., Chase, C., & Schwartz, D. L. (in press). Preparation for future learning with Teachable Agents. *Educational Technology Research and Design.*

Confrey, J. (1995). A theory of intellectual development: Part III. *For the Learning of Mathematics, 15*(2), 36-45.

Confrey, J. (this volume).

Driscoll, M. (1983). Research *within reach: Secondary school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.

Gibson, E. (1969). *Principles of perceptual learning and development*. New York: Appleton-Centrury-Crofts.

Glenberg, A. M., Gutierrez, T,, Levin, J. R., Japuntich, S., & Kaschak, M. P. (2004). Activity and imagined activity can enhance young children's reading comprehension. *Journal of Educational Psychology, 96*, 424-436.

Goldstone, R. L., & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *Journal of the Learning Sciences, 14*, 69-110.

Goodwin, C. (1994). Professional Vision. *American Anthropologist, 96 (3)*, 606-633.

Greeno, J., Smith, D. R., & Moore, J. L. (1993) Transfer of situated learning. In D. K. Detterman

& R. J. Sternberg (Eds.), *Transfer on trial: Intelligence, cognition, and instruction* (pp. 99-167). Norwood, NJ: Ablex.

Harp, S. F., & Mayer, R. E. (1998). How seductive details do their damage: A theory of cognitive interest in science learning. *Journal of Educational Psychology, 90*, 414-434.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition & Instruction, 14*, 251-283.

Kaminski, J. A., Sloutsky, V. M. & Heckler, A. F. (2008) The advantage of abstract examples in learning math. *Science, 320*, 454-455.

Kaminski, J., Sloutsky, V. M., & Heckler, A. (2009). Transfer of mathematical knowledge: The portability of generic instantiations. *Child Development Perspectives, 3,* 151–155.

Kapur, M. (2010). Productive failure in mathematical problem solving. *Instructional Science, 38(6)*, 523-550.

Kellman, P. J., Massey, C. M., & Son, J. Y. (2010). Perceptual learning modules in mathematics: Enhancing students' pattern recognition, structure extraction, and fluency. *Topics in Cognitive Science, 2*, 285-305.

Koedinger, K.R. & Nathan, M.J. (2004).The real story behind story problems: Effects of representations on quantitative reasoning./The Journal of the Learning Sciences, 13/ (2), 129-164.

Lajoie, S. P. (1998). *Reflections on Statistics*. Mahwah, NJ: Erlbaum.

Lakoff , G & Núñez. R. (2000). *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. New York: Basic Books.

Martin, T. (2009). A theory of physically distributed learning: How external environments and internal states interact in mathematics learning. *Child Development Perspectives, 3,* 140–144.

Martin, T., & Schwartz, D. L. (2005). Physically distributed learning: Adapting and reinterpreting physical environments in the development of the fraction concept. *Cognitive Science, 29,* 587-625.

Marton, F and Booth, S. (1997). *Learning and Awarenes*s. Mahwah, NJ: Lawrence Erlbaum.

Moyer, P. S. (2001) Are we having fun yet? How teachers use manipulatives to teach mathematics. *Educational Studies in Mathematics, 47*, 175–197.

Myles-Worsley, Johnston, W. A. and Simons, M. A. (1988). The influence of expertise on x-ray image processing. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 14*, 553-557.

Piaget, J. (1941/1952). *The Child's Conception of Number.* London: Routledge and Kegan Paul.

Post, T.R., (1981). Fractions: Results and Implementation from National Assessment. *Arithmetic Teacher, 28,* 26-31.

Resnick, L. (1983). Mathematics and science learning: A new conception. *Science, 220,* 477-478.

Reyna, V. F., & Brainerd, C. J. (2008). Numeracy, ratio bias, and denominator neglect in judgments of risk and probability. *Learning and Individual Differences, 18*, 89-107.

Schwartz, D. L. & Bransford, J. D. (1998). A time for telling. *Cognition & Instruction, 16*, 475-522.

Schwartz, D. L., Chase, C., and Bransford, J. D. (in press). Adpative Transfer: The tension of routines and novelty. *Educational Psychologist.*

Schwartz, D. L., Chase, C., Chin, D. B., & Oppezzo, M. (in review). *How telling too soon affects transfer: Practicing versus inventing with contrasting cases.*

Schwartz, D. L., & Martin, T. (2004). Inventing to prepare for learning: The hidden efficiency of original student production in statistics instruction. *Cognition & Instruction, 22*, 129-184.

Schwartz, D. L., Martin, T., & Pfaffman, J. (2005). How mathematics propels the development of physical knowledge. *Journal of Cognition and Development, 6*, 65-88.

Schwartz, D. L. & Moore, J. L. (1998). The role of mathematics in explaining the material world: Mental models for proportional reasoning. *Cognitive Science, 22*, 471-516.

Shen, J. (2006). Tools and task structures in modeling balance beam. In the *Proceedings of the 7th International Congress of the Learning Sciences*, 695-701.

Siegler, R. S. (1981). Developmental sequences within and between concepts. *Monographs of the Society for Research in Child Development, 46*(2), 1–74.

Sowell, E. J. (1989). Effects of manipulative materials in mathematics instruction. *Journal for Research in Mathematics Education, 20*, 498–505.

Thompson, P. W. (1994). Concrete materials and teaching for mathematical understanding. *Arithmetic Teacher, 41*, 556-558.

Uttal, D. H., Liu, L. L., & DeLoache, J. S. (1999). Taking a hard look at concreteness: Do concrete objects help young children learn symbolic relations? In C. S. Tamis-LeMonda (Ed.), *Child psychology: A handbook of contemporary issues* (pp. 177–192). Philadelphia, PA: Psychology Press.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes.* (M.

Cole, V. John-Steiner, S. Scribner, E. Souberman, Eds.) Cambridge, MA: Harvard University Press.

Table 1. Assumptions about student understanding and the relation of concrete and abstract materials during instruction.

Assumptions about initial understanding	**Concrete materials understood**	**Concrete materials not yet understood**
Symbols understood	**Practice** *Example:* Student finds the means and standard deviations of familiar kinds of distributions.	**Symbols → Concrete** *Example:* Student finds the mean and standard deviation of a novel kind of distribution.
Symbols not yet understood	**Concrete → Symbols** *Example:* Instructor uses a concrete data distribution to explain the meaning of the standard deviation formula.	**Co-Evolution** *Example:* Student uses symbols to invent an index that will characterize the shape of a distribution.

Table 2. Percentage of transitions between levels of feedback within and across problems.

Next Level

	Correct/ Incorrect	**Direction**	**Approximate Magnitude**	**Exact Magnitude**
Correct/incorrect	39%	15%	21%	21%
Direction	11%	--	22%	44%
Approximate Magnitude	18%	--	18%	55%
Exact Magnitude	--	--	--	83%

NB: Columns do not sum to 100%, because No-valid information codes are omitted.