Chapter 14

Cognitive Science Foundations of Integer Understanding and Instruction

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Abstract

This chapter considers psychological and neuroscience research on how people understand the integers, and how educators can foster this understanding. The core proposal is that new, abstract mathematical concepts are built upon known, concrete mathematical concepts. For the integers, the relevant foundation is the natural numbers, which are understood by reference to a mental number line (MNL). The integers go beyond the natural numbers in obeying the additive inverse law: for any integer x, there is an integer $-x$ such that $x + (-x) = 0$. We propose that practicing applying this law, such as when students learn that the same quantity can be added or subtracted from both sides of an equation, transforms the MNL. In particular, perceptual mechanisms for processing visual symmetry are recruited to represent the numerical symmetry between the integers x and $-x$. This chapter reviews psychological and neuroscience evidence for the proposed learning progression. It also reviews instructional studies showing that the hypothesized transformation can be accelerated by novel activities that engage symmetry processing compared to conventional activities around number lines and cancellation. Ultimately, these instructional insights can guide future psychological and neuroscience studies of how people understand the integers in arithmetic and algebraic contexts.

Keywords: integers, distance effect, intraparietal sulcus, mental number line, additive inverse law, symmetry processing, analog-x model, bundling hypothesis

Chapter 14

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When psychology and neuroscience ask how people understand mathematical concepts, they search for fundamental mechanisms of mind and brain. Studies from these fields have demonstrated that adults possess magnitude representations on which natural number concepts are constructed (Moyer & Landauer, 1967); have tracked the increasing precision of these representations over development (e.g., Sekuler, & Mierkiewicz, 1977; Xu & Spelke, 2005); and, have identified neural correlates of these representations (e.g., Pinel, Dehaene, Rivière, & Le Bihan, 2001). Central to cognitive science is the question of how these basic cognitive capacities are organized to understand culturally constructed number systems.

Education asks a different question. What experiences best support the learning of new, evermore abstract mathematical concepts? Research, for example, has investigated the ideal sequencing of concepts and procedures in mathematics instruction (Rittle-Johnson, Schneider, & Star, 2015; Rohrer & Taylor, 2007). It has also examined how to use concrete manipulatives to teach more abstract concepts (e.g., Martin & Schwartz, 2005). Ideally, the work of cognitive science can inform the educational enterprise of improving learning.

In this chapter, we develop the cognitive science foundations of how people understand integers and how these foundational insights contribute to instruction. The integers consist of a perceptually available number class, the natural numbers {0, 1, 2, …}, coupled with the much less perceptually obvious negative integers {-1, -2, …}. When walking in the woods, people can count the number of squirrels on their fingers, but they will not have an easy way to count the number of negative squirrels.

The integers are a relatively new human construction. The concept of negative numbers as debts arose as early as 250 BCE in China and 7th century India, but for much of history the idea of negative numbers was absurd, and the modern system of negative numbers did not arise until the 19th century (Gallardo, 2002; Hefendehl-Hebeker, 1991). The integers provide an excellent point of contact for psychology, neuroscience, and education, because they are an important abstract concept that students need to learn. They also represent a quantitative system that is culturally constructed. Unlike the perceptual sense of magnitude, which helps understand that 5 is bigger than 4, negative numbers do not exhibit an obvious mapping to basic perceptual abilities. Thus, they represent a test-bed for researchers from all three disciplines to study how an abstract mathematical concept can be nurtured from fundamental cognitive and perceptual-motor capacities.

14.1 A Learning Progression for Integer Understanding

How might one understand numerical expressions such as “-4”, questions about magnitude such as which is greater -4 or 3, and questions about arithmetic expressions such as “-4 + 3”? One intuition might be that people do so by reference to a mental number line (MNL), organized and oriented in the mind’s eye in the same way as physical number lines are organized and oriented in the world. Zero would be in the middle, negative integers on the left side, and positive integers on the right. We call this model *analog+* because it extends the well-established MNL for natural numbers (Moyer & Landauer, 1967; Sekuler, & Mierkiewicz, 1977).

An alternative intuition might be that negative integers are too abstract to represent directly, and that people reason about them using positive numbers and rules for manipulating the negative and positive signs. For example, to decide if -4 is less than -7, one might reason that

7 is greater than 4, but with negative numbers, one reverses the decision, so -4 is greater than -7. To decide if -4 is greater than 3, one might apply a rule that negative numbers are always less than positive numbers. We call this model *symbol+* under the assumption that mapping is via symbolic rules, and negative magnitudes are not accessed directly.

Recent research indicates both analog+ and symbol+ have merit. People obviously can reason about integers in these ways, as demonstrated by the fact that one can understand the verbal descriptions of each model in the preceding paragraphs. Surprisingly, however, adults appear to rely on yet a third model that lends more sophistication to their abilities to reason about integers. In the following, we describe this model and offer hypotheses for how it develops and how instruction can support it.

In doing so, we build on our earlier proposals (Blair, Tsang, & Schwartz, 2014; Schwartz, Blair, & Tsang, 2012) to develop a learning progression for how people come to understand abstract mathematical concepts such as the integers. This proposal is depicted in Figure 14.1. New mathematical concepts are built upon known mathematical concepts, but they can also incorporate additional perceptual primitives that provide structure not found in the original mathematical concepts.

For the integers, the relevant foundational concepts come from knowledge of the natural numbers. As previewed above, psychological and neuroscience evidence suggests that natural numbers are understood by reference to magnitude representations organized as an MNL. These representations support judgments such as deciding which of 1 and 9 is greater.

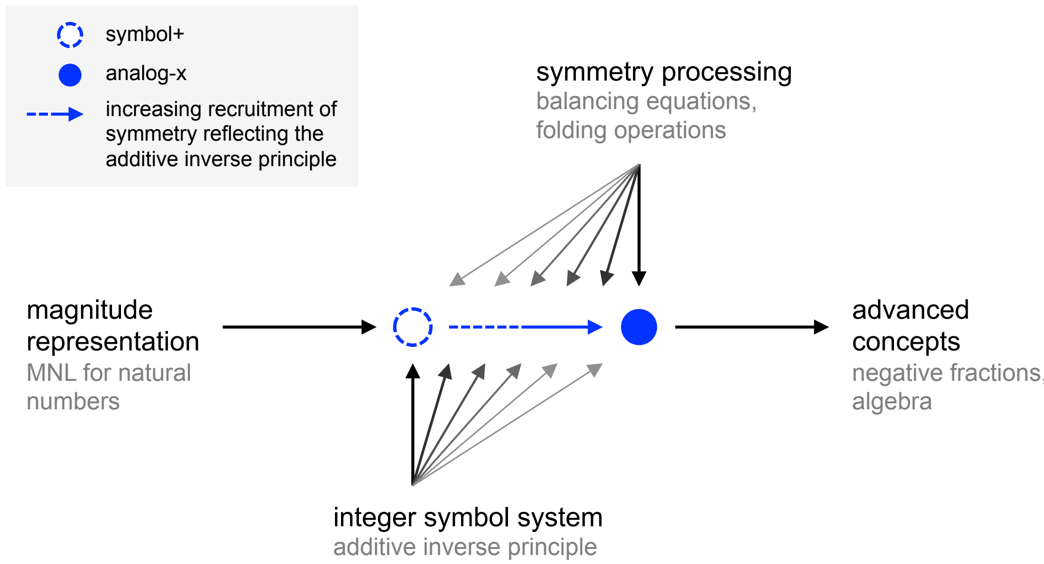

Figure 14.1 Learning progression for integer understanding. People understand the natural numbers using magnitude representations. Initially, they reason about integers directly, by using the rules of the defining symbol system, most notably the additive inverse principle (symbol+). Through experiences with balancing equations, they recruit symmetry processing, transforming their mental representation of integers to directly reflect the additive inverse principle (analog-x). The transformed representation supports learning of advanced mathematical concepts.

When people first learn about the integers, they reason about them using the rules of the governing symbol system, i.e., according to symbol+. This is not surprising: Conventional classroom instruction introduces procedures for handling this new, abstract number class by reference to the procedures for handling natural numbers – the more concrete number class that children have already mastered.

Children learn the integers, but the standard instruction does not capture the key law that creates the class of integers. This is the *additive inverse* law, which states that any integer plus its "inverse" equals zero: $x + -x = 0$. Our proposal is that as children learn algebra, they practice applying the additive inverse law in its colloquial form: the same quantity can be added or subtracted from both sides of an equation. This practice transforms their understanding of integers, extending the MNL for natural numbers "to the left" of zero, to also include the negative integers. Critically, this new MNL is *not* a simple extension of the positive number line as suggested by analog+, but rather a transformation that incorporates the *symmetry* between pairs of additive inverses x and $-x$ in a novel way. In doing so, it combines the mind's capacity for representing magnitudes with its capacity for processing symmetry. We call this transformed mental representation *analog-x*.

In the remainder of this chapter, we develop the case for the learning progression depicted in Figure 14.1. We begin with a review of psychological and neuroscience studies of how adults mentally represent the integers, and how this representation shifts over development. This research has primarily evaluated the analog+ and symbol+ models and found both wanting. We next introduce the analog-x model, which accounts for many of these challenging findings, and consider additional evidence for its proposals. We then selectively review classroom-based research, focusing on examples of symmetry-based instruction, which then feeds back to further inform our understanding of mental representations.

14.2 Cognitive and Developmental Science Studies of Integer Understanding

Our review of the literature begins with a consideration of some cognitive and developmental studies of how adults and children understand integers. Psychologists and neuroscientists utilize a set of standard laboratory paradigms for investigating the mental representation and processing of symbolic numbers, particularly with respect to how they relate to perceptual-motor primitives for comparing physical magnitudes. In this section, we selectively review studies that have used some of these paradigms to reveal how people understand negative integers. Our focus is on findings that distinguish the analog+ and symbol+ models, and that motivate the analog-x model developed in the next section.

14.2.1 Distance Effect

The comparison paradigm dominates studies of numerical cognition. In this paradigm, people are presented with two numbers and make a judgment about which one is greater (or lesser) as quickly as possible while maintaining high accuracy. Response times are then used to make inferences about underlying mental representations and processes. Moyer and Landauer (1967) found that when adults compare pairs of one-digit natural numbers, the farther apart the numbers, the faster the judgment (e.g., 1 vs. 9 is judged faster than 1 vs. 3). From this *distance effect*, they inferred that people understand one-digit natural numbers using magnitude representations. More precisely, people possess a MNL for natural numbers, organized and oriented in space with smaller numbers "on the left" and larger numbers "on the right".[1] When comparing which of two numbers is greater, they map them to points on the MNL and discriminate which point is "to the right". The farther apart the points, the easier the discrimination, and thus the faster the judgment. The distance effect has been extended to infants and children (Sekuler, & Mierkiewicz, 1977; Xu & Spelke, 2005), and to multi-digit natural numbers, rational numbers, and irrational numbers (Dehaene, Dupoux, & Mehler, 1990; Patel & Varma, in press; Schneider & Sigeler, 2010; Varma & Karl, 2013).

Recently, psychological researchers have used the comparison paradigm to investigate the mental representation of integers. Two kinds of comparisons have received the bulk of attention. For negative comparisons, where both numbers are negative integers, adults and

[1] Cultural differences may influence the left-right orientation of the number line, based on whether numbers are read from left-to-right or right-to-left in one's native language. However,

children show a distance effect. For example, they compare -1 vs. -9 faster than they compare -1 vs. -3 (Tzelgov, Ganor-Stern, & Maymon-Schreiber, 2009; Varma & Schwartz, 2011). This finding is consistent with analog+, which proposes that negative integers are represented as points on the extended MNL to the left of zero. The greater the distance between two points, the easier it is to discriminate which point is farther "to the right", just as it is when comparing natural numbers; hence the distance effect. This finding is also consistent with symbol+, which proposes that comparisons of negative integers are first mapped to comparisons of positive integers (e.g., which is greater, -1 vs. -9? → which is lesser, 1 vs. 9?); the positive integers are compared using the MNL for natural numbers (e.g., 1 < 9); and these judgments are mapped back to the negative integer domain (e.g., 1 < 9 → -1 > -9). It is the middle step, where the mapped positive integers are compared using the MNL, that produces the distance effect. Thus, negative comparisons cannot differentiate analog+ and symbol+ because both models predict a distance effect.

What can differentiate the two models are mixed comparisons, where one integer is negative and the other positive (e.g., -1 vs. 2 and -1 vs. 7). Analog+ proposes that the two integers are mapped to points on the extended MNL and discriminated, and therefore predicts a distance effect. By contrast, symbol+ proposes that the rule "positives are greater than negatives" is applied. Because this rule does not rely on magnitude representations, there should be no effect of distance.[2] These conflicting predictions mean that, in principle, the data can be used to choose between the two models. However, in practice, this has proven difficult. One difficulty is that relatively few psychological studies have looked for distance effects (or a lack thereof) for

[2] In addition to mixed comparisons, zero comparisons can also differentiate the analog+ and symbol+ models. These are comparisons where one of the two numbers is zero (e.g., -2 vs. 0). See Varma and Schwartz (2011) for further discussion.

mixed comparisons. Another difficulty is that those that have done so have found inconsistent results. Nevertheless, some inferences are possible.

No study has found a conventional distance effect for mixed comparisons, which would be consistent with analog+. Some studies have found no effect of distance, consistent with symbol+. For example, Tzelgov et al. (2009) found no effect of distance for mixed comparisons of the form -*x* vs. *y*, where the integers have different absolute values (e.g., -2 vs. 4); see also Ganor-Stern, Pinhas, Kallai, and Tzelgov (2010). Remarkably, other studies have found an *inverse* distance effect! Tzelgov et al. (2009) found an inverse distance effect for mixed comparisons of the form -*x* vs. *x*, where the integers have the same absolute value (e.g., -1 vs. 1 is judged faster than -4 vs. 4). Varma and Schwartz (2011) also found an inverse distance effect for mixed comparisons of the form *x* vs. *y* (e.g., -1 vs. 2 is judged faster than -1 vs. 7); see also Krajcsi and Igács (2010). These mixed findings limit the strength of the inferences that can be drawn about the mental representation and processing of negative integers. With this caveat in mind, the remainder of this chapter assumes that the inverse distance effect is "real" (although we note several other inconsistencies in the literature below and give reasons for them in the Conclusion).

14.2.2 SNARC Effect

Further evidence for people's mental representation of number comes from the Spatial-Numerical Response Codes (SNARC) effect. This is the finding that smaller numbers are associated with the left side of space and larger numbers with the right side of space, reflecting their respective locations on the MNL as conventionally oriented. This effect was first documented in a study where adults judged the parity of one-digit natural numbers (Dehaene, Bossini, & Giraux, 1993). Adults were faster to judge the parity of small numbers (e.g., 2) when the response (e.g., "even") was made on the left versus right side of space, and faster to judge the parity of large numbers (e.g., 9) when the response (e.g., "odd") was made on the right versus left side of space. The SNARC effect for one-digit natural numbers has been replicated many times (Gevers & Lammertyn, 2005). However, this effect extends inconsistently to other number classes such as multi-digit natural numbers and rational numbers (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Toomarian & Hubbard, 2018; Varma & Karl, 2013).

Analog+ and symbol+ agree in predicting that the SNARC effect extends to integers. However, they make different predictions regarding the form of this extension. Analog+ predicts a continuous SNARC effect, with negative integers responded to faster on the left versus right side of space and positive integers showing the opposite pattern. This is because it proposes that negative integers correspond to points "to the left" of zero on the MNL (and positive integers to points "to the right" of zero). By contrast, symbol+ predicts a piecewise SNARC effect, with negative integers showing an *inverse* SNARC effect (and positive integers a conventional SNARC effect). The inverse SNARC effect results because negative integers are mapped to positive integers before processing them (i.e., $-x \rightarrow |-x| \rightarrow x$). Thus, large negative integers are processed as small positive integers (e.g., -1 → 1) and small negative integers as large positive integers (e.g., -9 → 9), yielding an inverted SNARC effect.

That analog+ and symbol+ predict different SNARC effects means that, in principle, the data can be used to choose between them. Unfortunately, the literature is full of mixed results. Some studies have found the continuous SNARC effect predicted by analog+ (Fischer, 2003) whereas others have found the piecewise SNARC effect predicted by symbol+ (Fischer & Rottman, 2005). Shaki and Petrusic (2005) showed that these different findings are due in part to differences in methodology. They had adults make positive comparisons (e.g., 1 vs. 2) and negative comparisons (e.g., -1 vs. -2), holding the distance between each pair of numbers constant. When positive comparisons and negative comparisons were intermixed in the same block of trials, participants showed a continuous SNARC effect consistent with analog+. However, when these different comparison types were segregated in different blocks, participants showed the piecewise SNARC effect predicted by symbol+. This study suggests that adults possess multiple integer representations and choose among them based on task demands. We return to this flexibility in the Conclusion.

14.2.3 Number Line Estimation Task

The number line estimation (NLE) paradigm has also been used to investigate the mental representation of integers. In this paradigm, participants are presented with a number and a number line with only the endpoints labeled, and have to mark the position of the number on the number line with a pencil or computer pointer. This task was originally used with children and with natural numbers in the ranges 1-100 to 1-1000. Not surprisingly, the error in children's estimates decreases over development. The more interesting finding was that the pattern of errors also changes over development. The pattern for older children is veridical, with linearly spaced numbers. By contrast, the pattern for younger children is logarithmic, with exaggerated spaces between smaller numbers and compressed spaces between larger numbers (Siegler & Opfer, 2003). These developmental trends have been extended to rational numbers, whether expressed as fractions or decimal proportions. In both cases, children as young as 10 years old already make linear estimates, with error decreasing with further development into adulthood (Iuculano & Butterworth, 2011; Siegler, Thompson, & Schneider, 2011). Finally, for irrational numbers, adults make linear and accurate estimates of radical expressions such as $\sqrt{2}$ and $\sqrt{90}$ (Patel & Varma, in press).

Analog+ and symbol+ do not make strong predictions about performing the NLE task on integers, and how this performance changes over development. For this reason, we simply present some of the core findings. First, there appears to be a logarithmic-to-linear shift with development in the estimation of negative integers, one that parallels that for natural numbers. Brez, Miller, and Ramirez (2015) found evidence that second graders rely on logarithmically scaled representations when estimating numbers in the range -1000 – 0, just as they do when estimating numbers in the range 0 – 1000. This representation shifts over elementary school, and by fourth (and especially sixth) grade, children exhibit linear representations for both ranges. By middle school, children's estimates are linear in the much larger range -10000 – 0 and also in the combined range -1000 – 1000 (Young & Booth, 2015).

14.2.4 Neuroimaging Studies

Additional insight into the mental representation and processing of integers can be gained from neuroscience studies. We focus here on functional Magnetic Resonance Imaging (fMRI) studies that have utilized the comparison paradigm, as these are of greatest relevance to competitively evaluating the analog+ and symbol+ models.

Chassy and Grodd (2012) identified areas that show greater activation when adults make negative comparisons (e.g.. -3 vs. -2) versus positive comparisons (e.g., 5 vs. 4). One such area was the superior parietal lobule (SPL). This area is adjacent to the intraparietal sulcus (IPS), which prior studies have identified as a neural correlate of the MNL for natural numbers. Specifically, the IPS shows a neural distance effect when comparing natural numbers, with greater activation for harder near-distance comparisons (e.g., 1 vs. 2) than for easier far-distance comparisons (e.g., 1 vs. 9) (Pinel et al., 2001). The researchers interpreted activation of the SPL similarly, as evidence that negative integers also have magnitude representations.[3] This interpretation is consistent with the extended MNL representation proposed by analog+ but not with the mapping rules of symbol+.

Stronger evidence would be provided by an experiment that looked for distance effects and that included mixed comparisons. Blair, Rosenberg-Lee, Tsang, Schwartz, and Menon, (2012) provided such evidence in an fMRI study of adults who made positive, negative, and mixed comparisons of pairs of integers in which the distance varied systematically. Their results concerning positive vs. negative comparisons largely replicated those of Chassy and Grodd (2011). A finding of interest involved a representational similarity analysis. In this kind of analysis, the neural response patterns elicited by different stimuli are compared. The idea is that the more dissimilar the patterns for two stimuli, the more distinct the representations. The researchers focused on the IPS and the neural patterns elicited when people make near- vs. far-distance comparisons. They found that near- vs. far-distance comparisons elicited more distinct neural patterns for positive integers than for negative integers. This implies that positive integers may have a "sharper tuning" in IPS than negative integers. This finding is consistent with analog+, suggesting that negative integer magnitudes are less well-differentiated than positive integer magnitudes. With regard to mixed comparisons, this study did not find a behavioral effect of distance, but the neuroimaging data told a more complex story. No areas were more active for mixed comparisons than for negative comparisons or positive comparisons. This null result is

[3] SPL and IPS are also associated with visuospatial reasoning (e.g., Zacks, 2008). Thus, it is possible that they are recruited here not to process the magnitudes of positive integers and negative integers, but rather to process their symmetric relationship about zero. We consider the role of symmetry processing in integer understanding below, when describing the analog-x model.

inconsistent with symbol+, which predicts recruitment of areas in prefrontal cortex associated with rule application (i.e., "positives are greater than negatives").

Gullick, Wolford, and Temple (2012) conducted a study similar to Blair et al. (2012). The results were comparable overall, but one finding is worth highlighting. There was an inverse distance effect for mixed comparisons of the form $-x$ vs. y where both $-x < y$ and $|x| < |y|$ (e.g., -3 vs. 5). This was true behaviorally, with far-distance comparisons slower than near-distance comparisons, and this was also true neurally, with far-distance comparisons eliciting greater activation in IPS and SPL than near-distance comparisons. These inverse behavioral and neural distance effects are inconsistent with both analog+, which predicts conventional distance effects, and with symbol+, which predicts no effects of distance.

To summarize, these neuroimaging studies of adults provide limited insight into the representation and processing of negative integers. Negative comparisons elicit greater activation than positive comparisons in IPS and SPL, areas associated with the MNL and visuospatial processing (Blair et al., 2012; Chassy & Grodd, 2012; Gullick et al., 2012). In addition, negative comparisons do not elicit greater activation than positive comparisons in prefrontal areas associated with rule processing (Gullick et al., 2012). These findings can be interpreted as evidence for analog+ and against symbol+, respectively. However, neither of these models can explain the inverse distance effect that Gullick et al. (2012) found for (a subset of) mixed comparisons, both behaviorally and in the activations of IPS and SPL.

By contrast, the findings are clearer from the lone neuroimaging study that has investigated how children understand negative integers. Gullick and Wolford (2013) had 5th and 7th graders make negative comparisons and positive comparisons. The important finding was that for the 5th graders, negative comparisons elicited greater activation than positive comparisons in prefrontal areas associated with rule processing. For 7th graders, however, there was no such difference. This suggests that younger children reason according to symbol+. This also suggests that older children might have shifted to a new model of integer understanding, whether because of development, experience, or instruction. We consider a candidate model next.

14.3 Analog-x

The psychological and neuroscience literatures on integer understanding are small and in some cases inconsistent. Nevertheless, they support tentative inferences about the nature of the underlying mental representations and processes.

We begin with mixed comparisons, where people judge whether a positive integer or negative integer is greater, because this case provides the most leverage for choosing between possible models. Analog+ proposes that integers are understood with respect to an extended MNL, where negative integers are located "to the left" of zero. It predicts a standard distance effect for mixed comparisons, with far-distance pairs (e.g., -1 vs. 7) judged faster than near-distance pairs (e.g., -1 vs. 2). Because no study in the literature has found support for this prediction, analog+ can be ruled out. Symbol+ proposes that negative integers are not understood directly, by reference to magnitude representations, but rather indirectly, by applying rules. In particular, mixed comparisons are made by applying the rule "positive integers are greater than negative integers." Because this rule makes no reference to the magnitudes of the integers, symbol+ predicts no effect of distance. Varma and Schwartz (2011) found support for this prediction among 6th graders who had just learned about negative numbers. This makes sense if conventional instruction builds new procedures for working with integers on top of known procedures for working with natural numbers, which students have already mastered. Some studies of adults have also found support for this prediction (Ganor-Stern et al., 2010; Tzelgov et al., 2009).

However, our assessment is that adults likely reason according to a different model. This follows from numerous other studies of adults that have instead found an inverse distance effect for mixed comparisons (Gullick et al., 2012 for comparisons of the form $–x$ vs. y where both $–x < y$ and $|x| < |y|$; Krajcsi & Igács, 2010; Varma & Schwartz, 2011; Tzelgov et al., 2009, for mixed comparisons of the form $-x$ vs. x). In these studies, adults are faster to judge near-distance pairs (e.g., -1 vs. 2) than far-distance pairs (e.g., -1 vs. 7). The inverse distance effect is inconsistent with the predictions of the analog+ and symbol+ models, and raises the question of how adults understand the integers? We address it here by describing a third model and reviewing evidence for its key proposal, that adults understand integers by combing magnitude representations with symmetry processing.

14.3.1 Integer Understanding = Magnitude Representations + Symmetry Processing

The natural numbers coupled with the addition operation form a system that obeys the commutative law $x + y = y + x$, the associative law $(x + y) + z = x + (y + z)$, and the identity law $x + 0 = x$, with 0 the additive identity. Critically, the integers bring additional structure: they also obey the inverse law, which states that for every x, there is a corresponding $-x$ such that their sum is the identity $x + -x = 0$.

Extending one's understanding of number from the natural numbers to the integers requires understanding the additional structure brought by the inverse law. Initially, this understanding is explicit. When children first learn about the integers, they apply the governing laws in a deliberate and controlled manner to work with integers in arithmetic expressions and algebraic equations. This is one sense in which they reason according to symbol+. With development and experience, however, children's integer understanding shifts. They come to an implicit understanding of the integers, such that they no longer recruit rule-based processing as heavily. Rather, they gain an intuitive understanding of how integers can and cannot be manipulated in arithmetic and algebraic contexts. This raises the question of what it means to have an intuitive understanding of the integers, in particular to understand that additive inverse law that enriches them beyond the natural numbers.

Analog-x provides an answer to this question. It proposes that adults understand negative integers as they understand natural numbers, with reference to magnitude representations. That is, there is an MNL for integers. Critically, it is not the MNL proposed by analog+: it does not *extend* the MNL for natural numbers "to the left". Rather, it *reflects* the MNL for natural numbers to directly represent the inverse relationship between the pairs *-x* and *x*. In this way, analog-x combines the mind's capacity for representing magnitudes with its capacity for processing symmetry.

Figure 14.2 depicts the combination of magnitude and symmetry mechanisms proposed by analog-x. At the center is a reference axis that helps locate the natural number MNL and the negative integer MNL. The natural number MNL is shown above the reference axis. Its nonlinear form captures the psychophysical scaling of magnitude representations. The magnitude of a natural number is given by the height of the corresponding point above the reference axis. Natural numbers are compared in the usual way, by discriminating their magnitudes (i.e., heights). As the examples in Figure 14.2 show, the model predicts a distance effect for positive comparisons (i.e., 1 vs. 8 is more discriminable than 1 vs. 3).

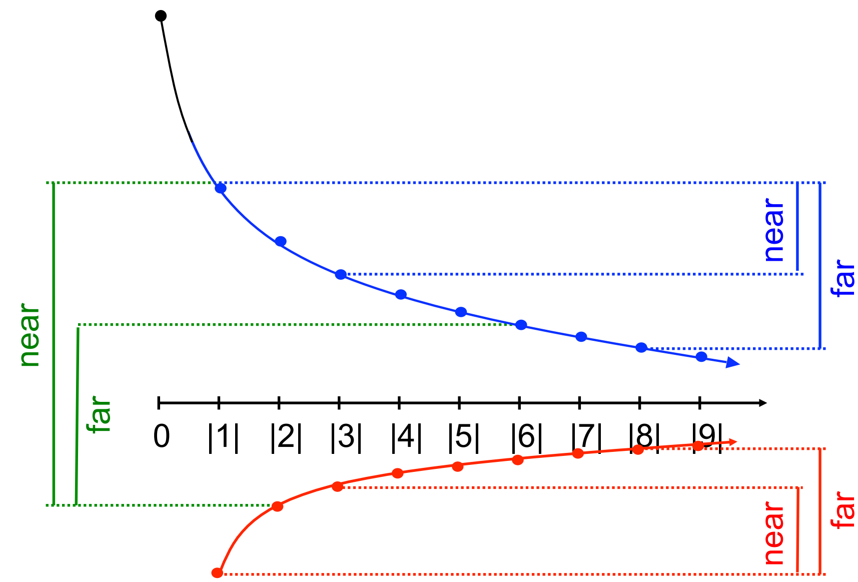

Figure 14.2 The analog-x model. The reference axis at the center helps locate the natural number MNL (above) and the negative integer MNL (below), which are reflections of each other. It predicts conventional distance effects for positive comparisons and for negative comparisons, as shown by the projections on the right. Critically, it predicts and inverse distance effect for mixed comparisons across the two MNLs, as shown by the projection on the left.

A new proposal is that the MNL for negative integers is a reflection of the MNL for natural numbers about the reference axis. This reflective organization has two important consequences. First, it directly models the inverse relationship between *–x* and *x*, in the vertical alignment of the corresponding points. In this way analog-x captures people's intuitive understanding of the additional structure that the integers bring over the natural numbers. Negative integers are compared in the same was as natural numbers, by discriminating the corresponding magnitudes (i.e., heights). The model predicts a distance effect for negative comparisons, as the examples in Figure 14.2 show (i.e., -1 vs. -8 is more discriminable than -1 vs. -3).

The second consequence of the reflective relationship between the natural number and negative integer MNLs concerns mixed comparisons. Specifically, this reflective relationship predicts the inverse distance effect observed by some researchers (Gullick et al., 2012 for comparisons of the form $-x$ vs. y where both $-x < y$ and $|x| < |y|$; Krajcsi & Igács, 2010; Varma & Schwartz, 2011; Tzelgov et al., 2009, for mixed comparisons of the form $-x$ vs. x). Positive and negative integers that are close together on the standard number line (e.g., -2 vs. 1), and thus hard to discriminate, correspond to magnitudes (i.e., heights) that are quite different in the analog-x representation, and thus easy to discriminate. The reverse is true for positive and negative integers that are far apart on the standard number line (e.g., -2 vs. 6): the corresponding heights in the analog-x representation are quite similar, and thus difficult to discriminate.[4]

14.3.2 Studies of Symmetry and Integer Processing

A novel proposal of analog-x is that the integer MNL encodes the additive inverse law using symmetry processing. Tsang and Schwartz (2009) tested this proposal in a behavioral study of adults. They developed an integer bisection paradigm where participants are presented with pairs of integers and have to name the midpoints as quickly as possible. They predicted that performance would be best for two cases where the symmetry of integers about 0 could be exploited. The first is for symmetric pairs of the form ($-x$, x), where the midpoint is 0. Computing the midpoint should be particularly easy because in analog-x, the corresponding points are vertically aligned to capture the additive inverse relationship between x and $-x$. The second case is for pairs of the form ($-x$, 0) and (0, x), where 0 – the point of symmetry – can be used to anchor midpoint estimation. They further predicted that symmetric processing would confer some advantage for pairs close to these two cases, e.g., (-6, 8) because it is almost symmetric, and (-1,

[4] The analog-x model shown in Figure 14.2 can be formalized and quantitatively fit to the data. See Varma and Schwartz (2011) for the details.

13) because it is almost anchored. Their results supported these predictions. Response times were fastest for bisections that were symmetric, anchored, or nearly so; see Figure 14.3a.

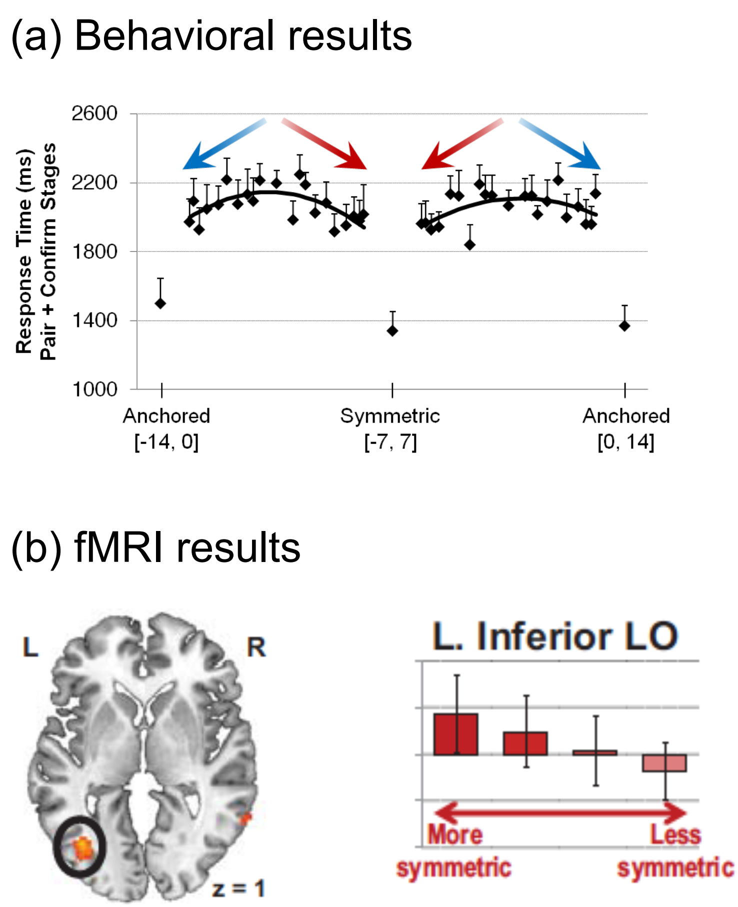

Figure 14.3 Integer bisection paradigm. (a) Bisection of integer pairs is privileged when the numbers are either more symmetric around 0 or more anchored to 0. (b) The greater the numerical symmetry of the pair, the greater the activation in left lateral occipital cortex, an area associated with processing of visual symmetry.

14.4 Instructional Studies and the Symmetry of Positive and Negative Integers

Analog-x is a model of how adults understand the integers. Its key proposal is that symmetry processing is recruited to represent that additive inverse law, resulting in a transformed MNL as shown in Figure 14.2. It is this symmetry that allows analog-x to predict the inverse distance effect for mixed comparisons and privileged performance on the integer bisection task for symmetrical and anchored pairs. In contrast, symbol+ provides a better characterization of children's understanding of integers. Behaviorally, children show no effect of distance (Varma & Schwartz, 2011), and neurally, they show increased recruitment of prefrontal areas associated with deliberate rule processing (Gullick & Wolford, 2013). This raises the question of the factors that drive the progression in how integers are understood, from applying symbolic rules to referencing a transformed MNL?

One hypothesized factor is learning algebra. This requires practicing applying the additive inverse law in its colloquial form: The same quantity can be added or subtracted from both sides of an equation. This practice could transform children's understanding of integers, restructuring their MNL to directly incorporate the symmetry between pairs of additive inverses x and $-x$; see Figure 14.1. Evidence for this developmental claim could come from a longitudinal study tracking changes in the integer representation over schooling. Unfortunately, no such study has been run to date.

Another perspective on how the integer representation changes over developments comes from instructional studies of how best to teach the integers to children. Some of these interventions have emphasized the use of standard number lines, and can be understood as consistent with analog+ (Hativa & Cohen, 1995; Moreno & Mayer, 1999; Schwarz, Kohn, & Resnick, 1993; Thompson & Dreyfus, 1988). Others have focused on teaching rules for manipulating negative numbers (e.g., the SR condition of Moreno and Mayer, 1999**),** consistent with the symbol+ model. Still others have focused directly on the additive inverse principle, using different colored chips or other discrete entities to represent positive and negative quantities, which cancel each other out (Bolyard, 2006; Liebeck, 1990; Linchevski & Williams, 1999; Streefland, 1996). A potential downside of these discrete cancellation-based approaches is that they do not emphasize order, and thus are isolated from linear magnitude representations of number (Bofferding, 2014).

Three recent studies have moved beyond instructional approaches aligned with analog+ and symbol+ or focused on the additive inverse principle in isolation. These studies have developed new approaches to instruction that focus on the symmetry of the positive and negative integers about zero, and as a result they are better aligned with analog-x.

14.4.1 Instructional Approaches Incorporating Symmetry

Two recent studies that have started from a conceptual analysis of the elements necessary to understand integers have derived instructional approaches that incorporate a focus on symmetry. Saxe, Earnest, Sitabkhan, Haldar, Lewis, and Zheng (2010) designed one part of their instruction around the task of marking the position of an integer on a standard number line, where other numbers might already be marked. They identified five principles necessary for successful performance. The fifth principle was understanding symmetry and absolute value: "For every positive number, there is a negative number that is the same distance from 0." (p. 440) Their instructional materials included problems that required reasoning about this symmetry, such as locating -150 on a number line where 0 and 150 were already marked. Their learning assessments included items measuring understanding of this symmetry, such as judging as correct or incorrect a number line where -1000 and 1000 were marked but were not equidistant from 0 (which was also marked), and providing a justification for why.

Bofferding (2014) developed and evaluated new instructional approaches for teaching the integers to first graders.[5] These approaches derived in part from a conceptual analysis of what it means to understand the integers, which revealed three meanings of the "-" sign. The first and second meanings are familiar: as a mark distinguishing negative integers from positive integers (e.g., -7 vs. 7) and as the name of the subtraction function (e.g., 9 – 3). The third meaning had been previously overlooked in the education literature: as the name of the "symmetric function" for "taking the opposite" (e.g., -(7) = -7). This study also revealed the roles symmetry plays in the mental models children have for the integers. Only the most sophisticated of these models represents that positive integers and negative integers are symmetric about zero. In addition, only these models correctly distinguish the values versus magnitudes of negative integers (e.g., 8 < 9 but -8 > -9), which is critical for making "more" versus "less" (and "high" versus "low") judgments of negative integers.

In these studies, symmetry is thoughtfully incorporated into the instruction and models of student learning. An important limitation is that the value of symmetry for learning is not tested directly.

14.4.2 An Instructional Study Directly Comparing Symmetry to other Approaches

We see convergence in the psychological, neuroscience, and mathematics education literatures that symmetry plays a critical role in what it means to understand the integers. A study that builds on this convergence is Tsang, Blair, Bofferding, and Schwartz (2015), which directly compared an instructional approach that incorporated symmetry to more traditional number line and cancellation approaches. The instructional approaches were built around three manipulatives embodying different underlying models. The "jumping" approach modeled arithmetic operations

[5] This study is notable in testing children much younger than those in prior psychological and educational studies.

as movements along an extended number line (Figure 14.4a); it corresponds to analog+. The "stacking" approach modeled arithmetic operations on the cancellation of discrete items (Figure 14.4b). The "folding" approach combined elements of jumping (i.e., directed magnitudes) and stacking (i.e., cancellation) (Figure 14.4c). What is novel about this approach is that adding or subtracting integers requires bringing the two operands into alignment using symmetric processing.

Post-test measures found substantial evidence for the efficacy of the folding approach, and thus for the use of symmetry. When estimating the position of a negative integer on a number line where the corresponding positive integer was already marked, the folding group was most likely to use a symmetry strategy, which was associated with more accurate performance. More importantly, the folding group performed best on far-transfer problems such as estimating the position of negative fractions on number lines and solving missing operand problems (e.g., 1 + –4 = [ ] + –2), which had been not covered in class. These far-transfer findings are evidence for the analog-x proposal that symmetry is particularly important when students learn pre-algebra and must apply the additive inverse law to manipulate equations.

The results of Tsang et al. (2015) suggest that including symmetry in integer instruction allows learners to generalize to solve new types of negative number problems that they had not directly been taught, including those that focus on the additive inverse property. These findings bring useful questions back to the study of mental representations of number. For example, there are relatively few neuroimaging studies of integer processing in general, and even fewer where the participants are children. How do different instructional approaches affect children's neural representations of integers as they become more fluent? Does an instructional approach that focuses on symmetry and the additive inverse property increase the recruitment of brain areas associated with visual symmetry, even when learners are reasoning about symbolic numbers?

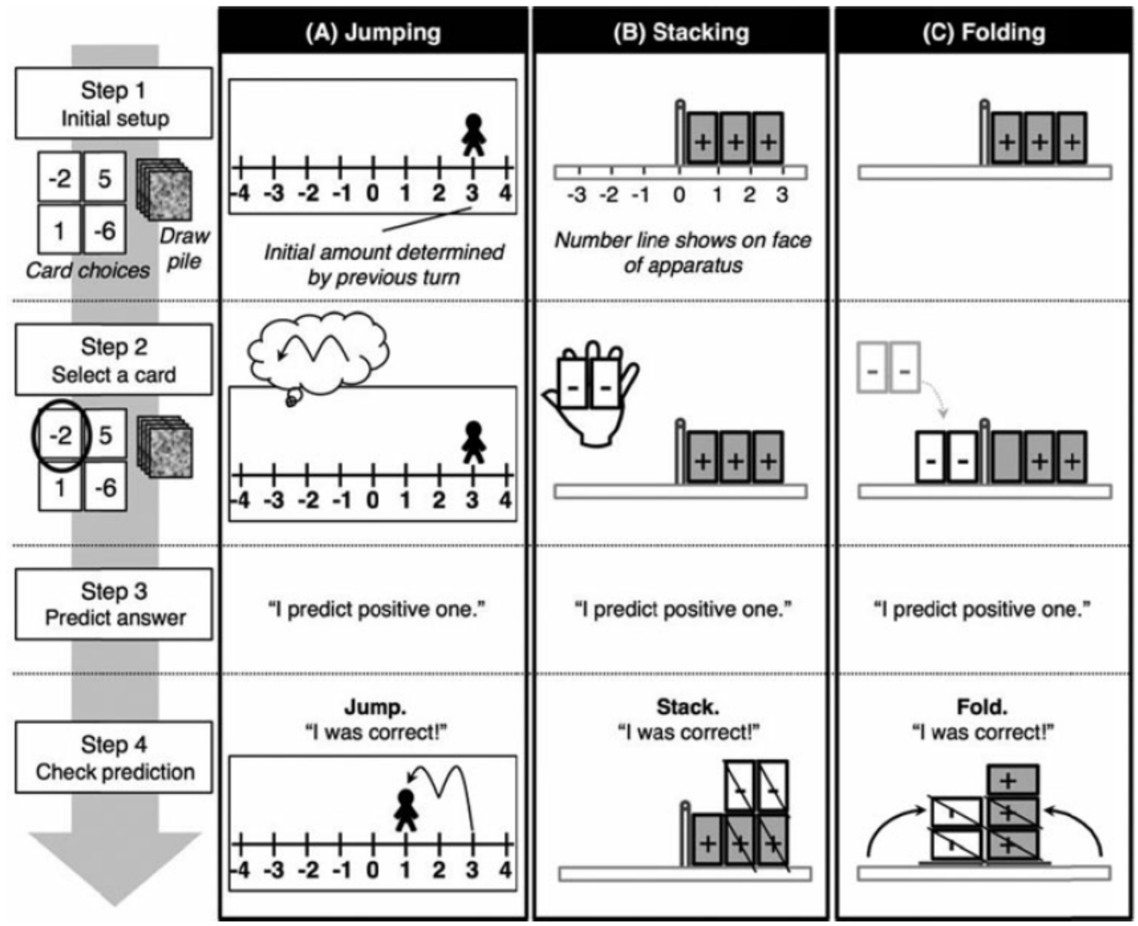

Figure 14.4 Actions taken by students when evaluating the equation 3 + –2 = 1 – in the (a) jumping, (b) stacking, and (c) folding instructional conditions. These correspond to the analog+, symbol+, and analog-x, respectively. (Note. From "Learning to 'see' less than nothing: Putting perceptual skills to work for learning numerical structure," by J. M. Tsang, K. P. Blair, L. Bofferding, and D. L. Schwartz, 2015, *Cognition and Instruction*, 33, p. 167. Copyright 2015 by Taylor & Francis. https://www.tandfonline.com/toc/hcgi20/current Reprinted with permission.)

14.5 Conclusion

This chapter has considered how adults understand an abstract mathematical concept, the integers, and how educators can foster this understanding in children. It has built a corridor of explanation from neuroimaging data to response times to hands-on activities in the classroom. The result is a clearer picture of how magnitude representations and symmetry processing support integer understanding, and how these capacities are coordinated and integrated through learning.

Our first proposal is that acquiring a new, abstract mathematical concept requires mastering the governing symbol system. More novel is our second proposal: mastery enriches the mental representation of known concepts to reflect the unique properties of the new concept, and it does so by recruiting additional perceptuo-motor capacities. In this way, people can build intuition for ideas quite far from perceptual-motor experience (Blair et al., 2014; Schwartz et al., 2012). Specifically, analog-x makes the surprising claim that the MNL for natural numbers is transformed through symmetry processing to directly encode that $-x$ and x are additive inverses. We speculate that this transformation is accelerated when students learn algebra, and practice applying the additive inverse law in its colloquial form ("the same quantity can be added to or subtracted from both sides of the equal sign") to manipulate equations. It is an open question whether this transformation can be accelerated further, for example by developing instructional activities where younger children coordinate magnitude representations and symmetry processing of integers. The folding condition of Tsang et al. (2015) offers initial evidence that this might be possible.

Our review began with neuroscience and psychological studies and progressed towards educational studies. We end by considering a path less often trodden: how education can inform psychology and neuroscience. Educational research can guide future lab studies of how analog-x (and analog+ and symbol+) scale to arithmetic and algebraic contexts. For example, there are few psychological studies of how people understand arithmetic operations on integers (e.g., Prather & Alibali, 2008), and the neural correlates of this understanding (e.g., Gullick & Wolford, 2014). By contrast, there is an extensive mathematics education literature on different approaches for teaching integer arithmetic (Hativa & Cohen, 1995; Liebeck, 1990; Linchevski & Williams, 1999; Moreno & Mayer, 1999; Saxe et al., 2010; Schwarz et al., 1993; Streefland, 1996; Thompson & Dreyfus, 1988). This asymmetry represents an opportunity for psychological and neuroscience research, as many of the phenomena that have been documented in the classroom merit further study in the lab. One example is Bofferding's (2014) proposal that to understand the integers is to understand three meanings of the "-" sign, including its easily overlooked meaning as a "symmetric function" for reversing the sign of an integer expression. Another example is the Tsang et al. (2015) finding that understanding the symmetric organization of positive integers and negative integers about zero is associated with better performance on pre-algebra problems demanding sensitivity to the meaning of the "=" sign (i.e., missing operand problems). What mental and neural mechanisms undergird understanding the "-" sign as a "symmetric function" and pre-algebraic reasoning about integers?

In addition, mathematics education research can potentially reframe how we understand the inconsistent results of some of the psychological studies reviewed above. This was true for the distance effect and the SNARC effect, with different studies finding evidence consistent with the three different models of integer understanding (i.e., analog+, symbol+, and analog-x). These inconsistencies are deeply problematic for psychologists and neuroscientists because they make it impossible to choose between competing models, and ultimately to make scientific progress.

The conventional explanation for mixed findings is noise in the signal: the samples are too small, the methods are too varied, and so on. Mathematics education research offers a different perspective on this heterogeneity. The participants in these studies learned about the integers in classrooms spread across the United States and indeed the world. We have seen that different instructional approaches are aligned with the three different models of integer understanding. Thus, it is possible that some of the inconsistencies observed in psychological studies are not the product of noise in the data or even individual differences in basic cognitive abilities. Rather, they may be the product of instructional differences. Understanding this systematic variation is a goal for future research.

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